An Analytical Solution of the Schrödinger Equation for a Rectangular Barrier with Time-Dependent Position

Tae Jun Park

Department of Chemistry, Dongguk University, Seoul 100-715, Korea

Received November 12, 2001

Keywords : Kramers-Henneberger transformation, Time-dependent, Rectangular barrier.

An analytical solution for the Schrödinger equation with time-dependent potential has been investigated extensively over past decades. In addition to its own mathematical interest, this problem has wide applications in many areas of physics, such as laser-induced dynamics, the motion of Paul trap ions,[1] and semiconductor physics.[2] Only systems with time-dependent potentials that are constant, linear, and quadratic in x are known to be analytically solved.[3]

For these problems, the well known methods for analytical wave functions are the famous invariant operator approach,[4] the propagator method,[5] and the time-space transformation method.[6] In general, systems with potentials of $V(x, t) = f(t)x^2 + g(t)x + h(t)$ has been solved exactly by these methods[7]. Among these systems, rectangular potentials with time-dependent height or depth are quite simple to solve.[8] A rectangular barrier with time-dependent position is, however, much more complex and the Schrödinger equation has not yet been solved analytically, although Moiseyev[9] studied the problem approximately by averaging the potential in time and by treating it as a time-independent bound system.

In the present work, we obtain the exact solution for the rectangular barrier whose position is oscillating in time. We use the Kramers-Henneberger transformation[10] which is a particular form of time-space transformation technique.

The Hamiltonian for the rectangular barrier with oscillating position is chosen as[9]

$$H(x, t) = \frac{p^2}{2m} + V(x, t), \tag{1}$$

where

$$V(x, t) = \begin{cases} V_0, \text{ if } |x + \alpha_0 \cos \omega t| < \dfrac{a}{2} \\ 0, \text{ elsewhere} \end{cases} \tag{2}$$

The position of the barrier oscillates with the frequency $\omega = 2\pi/T$ so that at $t = 0$ the barrier is centered at $x = -\alpha_0$, and at $t = T/2$ its center is at $x = +\alpha_0$. The Hamiltonian with the potential $V(x, t)$ of eq. (2) is obtained from $H = p^2/2m + V(x) + E_0 x\cos\omega t$ by Kramers-Henneberger transformation,[10] where $\alpha_0 = E_0/m\omega^2$. This Hamiltonian represents the system under the field $E_0 x\cos\omega t$.

If we introduce a new variable, $\xi(x, t) = x + \alpha_0\cos\omega t$, following the Kramers-Henneberger transformation,[10] the time-dependent wave function of the system, $\Psi(x, t)$, can be written as follows[3]

$$\Psi(x, t) = e^{-\frac{iEt}{\hbar}}\phi(\xi, t)\chi(x, t), \tag{3}$$

where E is a constant parameter which could be the energy of the system. Inserting $\Psi(x, t)$ of eq. (3) into time-dependent Schrodinger equation and changing x to ξ, we have

$$-\frac{\hbar^2}{2m}\left[x\frac{\partial^2\phi}{\partial\xi^2} + 2\frac{\partial\phi}{\partial\xi}\frac{\partial\chi}{\partial x} + \phi\frac{\partial^2\chi}{\partial x^2}\right] + V(\xi)\phi\chi$$
$$= i\hbar\left[\chi\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\phi}{\partial t}\right) + \phi\left(-\frac{iE}{\hbar}\chi + \frac{\partial\chi}{\partial t}\right)\right]. \tag{4}$$

Since the potential $V(\xi)$ in eq. (4) does not depend on t explicitly, $\phi(\xi, t)$ would be a time-independent solution if the following relation is satisfied:

$$-\frac{\hbar}{2m}\left[2\frac{\partial\phi}{\partial\xi}\frac{\partial\chi}{\partial x} + \phi\frac{\partial^2\chi}{\partial x^2}\right] = i\left[\chi\frac{\partial\xi}{\partial t}\frac{\partial\phi}{\partial\xi} + \phi\frac{\partial\chi}{\partial t}\right]. \tag{5}$$

Eq. (4) then becomes

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi^2} + V(\xi) - E\right)\phi(\xi, t) = i\hbar\frac{\partial\phi(\xi, t)}{\partial t}. \tag{6}$$

Solutions of eq. (6) would be $e^{\pm c_1\xi}$, where $c_1 = ik$ or χ $(k = \sqrt{2mE}/\hbar)$, $(\kappa = \sqrt{2m(V_0 - E)}/\hbar)$, depending on the region of x.

Substituting $\partial\xi/\partial t = -(p(t)/m)$ and $\phi(\xi) = e^{c_1\xi}$ into eq. (5) and then rearranging it, we have

$$\frac{\hbar}{2m}\frac{\partial^2\chi}{\partial x^2} + \frac{\hbar c_1}{m}\frac{\partial\chi}{\partial x} - i\frac{c_1 p(t)}{m}\chi = -i\frac{\partial\chi}{\partial t}. \tag{7}$$

To determine the solution, we factorize $\chi(x, t)$ as $\chi(x, t) = u(t)\upsilon(x)$ since the eq. (7) is not coupled in x and t. Inserting $\chi(x, t)$ into eq. (7) and then dividing both sides by $u(t)\upsilon(x)$, we obtain,

$$\frac{\hbar}{2m}\frac{1}{\upsilon}\frac{d^2\upsilon}{dx^2} + \frac{\hbar c_1}{m}\frac{1}{\upsilon}\frac{d\upsilon}{dx} = -i\left(\frac{1}{u}\frac{du}{dt} - \frac{c_1 p(t)}{m}\right). \tag{8}$$

Since the left-hand side is a function of x only, while the right-hand side is a function of t, we let both sides equal to c_2 which is a constant. Thus we have $u(t)$ as given below,

$$u(t) = e^{ic_2 t + \frac{c_1}{m}\int^t p(t')dt'} = e^{ic_2 t - c_1\alpha_0\cos\omega t}. \tag{9}$$

The left-hand side would be an ordinary second-order differential equation for $\upsilon(x)$ as,

$$\frac{\hbar}{2m}\frac{d^2\upsilon}{dx^2}+\frac{\hbar c_1}{m}\frac{d\upsilon}{dx}-c_2\upsilon = 0. \tag{10}$$

Inserting $\upsilon(x) = e^{\lambda(x)}$ into eq. (10), we obtain the equation for $\lambda(x)$ given as,

$$\frac{\hbar}{2m}\left[\frac{d^2\lambda}{dx^2}+\left(\frac{d\lambda}{dx}\right)^2\right]+\frac{\hbar c_1}{m}\frac{d\lambda}{dx}-c_2 = 0. \tag{11}$$

If we define $d\lambda/dx = w(x)$ and insert it into eq. (11), we finally have the first-order differential equation for $w(x)$ as given below,

$$\frac{\hbar}{2m}\frac{dw}{dx} = c_2-\frac{\hbar c_1}{m}w-\frac{\hbar}{2m}w^2, \tag{12}$$

which can be easily solved by integrating the equation given as,

$$\left(\frac{2m}{\hbar}c_2-2c_1w-w^2\right)^{-1}dw = dx. \tag{13}$$

Integrating eq. (13), we would have

$$x = -\frac{2}{\sqrt{-\Delta}}\tanh^{-1}\left(-\frac{2(c_1+w)}{\sqrt{-\Delta}}\right), \quad \Delta<0$$
$$= \frac{2}{\sqrt{\Delta}}\tan^{-1}\left(-\frac{2(c_1+w)}{\sqrt{\Delta}}\right), \quad \Delta>0, \tag{14}$$

where $\Delta=-4(2m/\hbar\, c_2+c_1^2)$. Determining $w(x)$ from eq. (14) and integrating it again, we have $\lambda(x)$ as given below,

$$\lambda(x) = \ln\left[\cosh\left(-\frac{\sqrt{-\Delta}}{2}x\right)\right]-c_1x, \quad \Delta<0$$
$$= \ln\left[\cos\left(\frac{\sqrt{\Delta}}{2}x\right)\right]-c_1x, \quad \Delta>0. \tag{15}$$

From $\upsilon(x) = e^{\lambda(x)}$, we get

$$\upsilon(x) = \cosh\left(-\frac{\sqrt{-\Delta}}{2}x\right)e^{-c_1x}, \quad \Delta<0$$
$$= \cos\left(\frac{\sqrt{\Delta}}{2}x\right)e^{-c_1x}, \quad \Delta>0. \tag{16}$$

Thus we have $\chi(x, t)$ as

$$\chi(x,t) = e^{ic_2t-\alpha_0c_1\cos\omega t}\cosh\left(-\frac{\sqrt{-\Delta}}{2}x\right)e^{-c_1x}, \quad \Delta<0$$
$$= e^{ic_2t-\alpha_0c_1\cos\omega t}\cos\left(\frac{\sqrt{-\Delta}}{2}x\right)e^{-c_1x}, \quad \Delta>0. \tag{17}$$

Inserting $\chi(x, t)$ from eq. (17) and $\phi(\xi, t)$ which is $e^{\pm c_1\xi}$ into eq. (3), we can exactly determine $\Psi(x, t)$ for the system of rectangular barrier with the oscillating position.

Acknowledgment. This work is supported by the Dongguk University research fund. The author dedicates this article to Professor Kyung Hoon Jung.

References

1. Paul, W. *Rev. Mod. Phys.* **1990**, *62*, 531.
2. Roy, D. K. *Quantum Mechanical Tunneling And Its Application*; World Scientific: Singapore, 1986.
3. Truscott, W. S. *Phys. Rev. Lett.* **1993**, *70*, 1900.
4. Lewis, Jr., H. R. *J. Math. Phys.* **1968**, *9*, 1976.
5. Yeon, K. H.; Kim, D. H.; Um, C. I.; George, T. F.; Pandey, L. N. *Phys. Rev. A* **1997**, *55*, 4023.
6. Feng, M. *Phys. Rev. A* **2001**, *64*, 034101-1.
7. Truax, D. R. *J. Math. Phys.* **1982**, *23*, 43.
8. Wagner, M., *Phys. Rev. B* **1994**, *49*, 16544; Wagner, M. *Phys. Rev. Lett.* **1996**, *76*, 4010.
9. Vorobeichik, I.; Lefebvre, R.; Moiseyev, N. *Europhys. Lett.* **1998**, *41*, 111.
10. Henneberger, W. C. *Phys. Rev. Lett.* **1968**, *21*, 838.