

GROUP SOLUTION FOR UNSTEADY BOUNDARY LAYER FLOW OF A MICROPOLAR FLUID NEAR THE REAR STAGNATION POINT OF A PLANE SURFACE IN A POROUS MEDIUM

M. M. ABD-ELAZIZ[†] and S. E. AHMED[‡]

[†] Faculty of Science, Department of Mathematic, South Valley University, Qena, Egypt
Mohamed_Aziz@yahoo.com

[‡] Faculty of Science, Department of Mathematic, South Valley University, Qena, Egypt
samehelsayedahmed@yahoo.com

Abstract — The transformation group theoretic approach is applied to the system of equations governing the unsteady boundary layer flow of a micropolar fluid near the rear stagnation point of a plane surface in a porous media. The application of a two-parameter group reduces the number of independent variables by two, and consequently the system of governing partial differential equations with boundary conditions reduces to a system of ordinary differential equations with appropriate boundary conditions. The possible form of potential velocity U_e is derived in steady and unsteady cases. The family of ordinary differential equations has been solved numerically using a fourth-order Runge-Kutta algorithm with the shooting technique. The effect of varying parameters governing the problem is studied.

Keywords— Micropolar fluid, stagnation point, porous medium.

I. INTRODUCTION

Boundary layers of non-Newtonian fluids have received considerable attention in the last few decades. Boundary layer theory has been applied successfully to various non-Newtonian fluids models. One of these models is the theory of micropolar fluids introduced by Eringen (1964), Eringen (1966), Eringen (1977). In this theory, the micropolar fluid exhibits the microrotational effects and micro-inertia. The difficulty of the study of such fluid problem is the paucity of boundary conditions and the existence of deformable microelements as well as the time as the third independent variable. Many attempts were made to find analytical and numerical solutions by applying certain special conditions and using different mathematical approaches. Lok *et al.* (2003a) used the Keller-box method in conjunction with the Newton's linearization technique to study the unsteady boundary layer flow of a micropolar fluid near the rear stagnation point of a plane surface. Also, Lok *et al.* (2003b) studied the unsteady boundary layer flow of a micropolar fluid near the forward stagnation point of a plane surface by using the Newton's linearization technique of Keller-box method. Seshadri *et al.* (2002) used the implicit finite difference scheme to study the unsteady mixed convection flow in the stagnation region

of a heated vertical plate due to impulsive motion.

On the other hand, studies with group method were used by Helal and Abd-el-Malek (2005), Abd-el-Malek *et al.* (2004). The mathematical technique which used in the present analyses is the two-parameter group transformation leads to a similarity representation of the problem. A systematic formalism is presented for reducing the number independent variables in systems which consist, in general, of a set partial differential equations and auxiliary conditions (such as boundary and/or initial conditions).

In the present work we consider the unsteady boundary layer flow of a micropolar fluid near the rear stagnation point of a plane surface in a porous media. The governing boundary layer equations have been transformed to ordinary differential equations via group analysis and these have been solved numerically using shooting technique. The effects of varying parameters governing the problem were studied.

II. MATHEMATICAL ANALYSIS

Let us consider the development of the two-dimensional boundary layer flow of a micropolar fluid near the rear stagnation point of a plane surface in a porous medium. The fluid which occupies a semi-infinite domain bounded by an infinite plane and remains at rest for time $t < 0$ and starts to move impulsively away from the wall at $t = 0$. In our analysis, rectangular Cartesian coordinates (x, y) are used in which x and y are taken as the coordinates along the wall and normal to it, respectively. The flow configuration is shown schematically in Fig. 1.

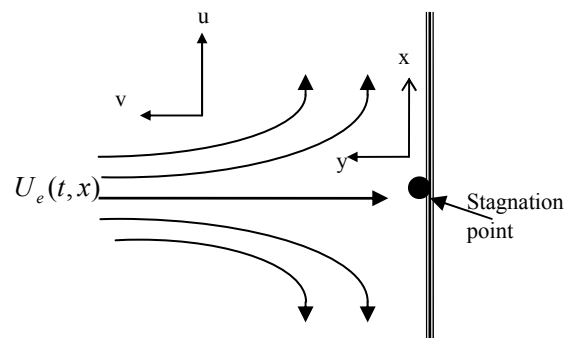


Fig.1. Physical model of the problem.

The boundary layer equations governing the unsteady flow of a micropolar fluid with constant physical properties are (Lok *et al.*, 2003a)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_e \frac{dU_e}{dx} \tag{2}$$

$$+ \left(\nu + \frac{k}{\rho} \right) \frac{\partial^2 u}{\partial y^2} + \frac{k}{\rho} \frac{\partial N}{\partial y} + \frac{\nu}{k_1} (U_e - u), \tag{3}$$

$$\frac{\partial N}{\partial t} + u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} = \frac{\gamma}{\rho j} \frac{\partial^2 N}{\partial y^2} - \frac{k}{\rho j} \left(2N + \frac{\partial u}{\partial y} \right), \tag{4}$$

$$\frac{\partial j}{\partial t} + u \frac{\partial j}{\partial x} + v \frac{\partial j}{\partial y} = 0. \tag{5}$$

In the above equations u and v are the components of fluid velocity in the x and y directions, respectively, U_e is the uniform stream velocity, N is the component of microrotation, ρ is the density, k is the vortex viscosity, γ is the spin-gradient viscosity, j is the microinertia density, ν is the kinematic viscosity and k_1 is the permeability of the porous medium.

The remaining equation is to be solved subject to the boundary and initial conditions

$$t < 0 : u(x, y, t) = 0, v(x, y, t) = 0, N(x, y, t) = 0, \tag{6}$$

$$t = 0 : u(t, x, \infty) = U_e(t, x), N(t, x, \infty) = 0,$$

$$t > 0 \begin{cases} u(x, y, t) = v(x, y, t) = N + \beta \frac{\partial u}{\partial y} = 0 & \text{at } y = 0, \\ u \rightarrow U_e(t, x), N \rightarrow 0 & \text{as } y \rightarrow \infty. \end{cases}$$

where β is a constant and $0 \leq \beta \leq 1$. The case $\beta=0$, which indicates $N=0$, represents concentrated particle flow in which the microelements close to the wall surface are unable to rotate. The case $\beta=1/2$ indicates to the vanishing of antisymmetric part of the stress tensor and denoted weak concentrations. The case $\beta=1$ is used for the modelling of turbulent boundary layer flows. We shall consider here only the value of $\beta=0$ and $\beta=1/2$.

At this point, we introduce the dimensionless stream function $\psi(x, y, t)$ such that $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$. Equations (2) and (3) become

$$\frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U_e \frac{dU_e}{dx} \tag{7}$$

$$+ \nu \left[(1 + \Delta) \frac{\partial^3 \psi}{\partial y^3} + \Delta \frac{\partial N}{\partial y} + \frac{1}{k_1} \left(U_e - \frac{\partial \psi}{\partial y} \right) \right],$$

$$\frac{\partial N}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial N}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial N}{\partial y} = \lambda \frac{\partial^2 N}{\partial y^2} \tag{8}$$

$$- \sigma \left(2N + \frac{\partial^2 \psi}{\partial y^2} \right),$$

with the boundary conditions

$$y = 0 : \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} = N + \beta \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{9}$$

$$y \rightarrow \infty : \frac{\partial \psi}{\partial y} = U_e, \frac{\partial \psi}{\partial x} = N = 0,$$

where $\Delta = k/\mu$ (coupling constant), $\lambda = \gamma/\rho j$ (microrotation parameter) and $\sigma = k/\rho j$ (dimensionless material parameter).

III. METHOD OF SOLUTION

Group analysis is the only rigorous mathematical method to find all symmetries of a given differential equation and no ad hoc assumptions or a prior knowledge of the equation under investigation is needed. The boundary layer equations are especially interesting from a physical point of view because they have the capacity to admit a large number of invariant solutions *i.e.* basically analytic solutions. In the present context, invariant solutions are meant to be a reduction to a simpler equation such as an ordinary differential equation (ODE). The non-linear character of the partial differential equations governing the motion of a fluid produces difficulties in solving the equations. In the field of fluid mechanics, most of the researchers try to obtain the similarity solutions in such cases.

A. Group Formulation of the Problem

In this section, two parameter transformation group is applied to the system of Eqs. (6) and (7) with the boundary conditions (8). The system of equations reduces to a system of ordinary differential equation in a single independent variable with appropriate boundary conditions. The procedure is initiated with the group G , a class of two parameter group of the form

$$G : \bar{S} = c^S(a_1, a_2)S + k^S(a_1, a_2), \tag{10}$$

where S stands for x, y, t, ψ, U_e and N the c'_s and k'_s are real valued and at least differentiable in their arguments (a_1, a_2) . Thus, in the notation of the given representation, the present analysis is initiated with a class C_G of two-parameter transformation groups with the form

$$G = \begin{cases} \bar{x} = c^x(a_1, a_2)x + k^x(a_1, a_2) \\ \bar{y} = c^y(a_1, a_2)y + k^y(a_1, a_2) \\ \bar{t} = c^t(a_1, a_2)t + k^t(a_1, a_2) \\ \bar{\psi} = c^\psi(a_1, a_2)\psi + k^\psi(a_1, a_2) \\ \bar{N} = c^N(a_1, a_2)N + k^N(a_1, a_2) \\ \bar{U}_e = c^{U_e}(a_1, a_2)U_e + k^{U_e}(a_1, a_2), \end{cases} \tag{10}$$

which possesses complete sets of absolute invariant $\eta(x, y, t)$ and $g_i(x, y, t, N, U_e, \psi)$, $i=1,2,3$ where g_i are the three absolute invariants corresponding to N, ψ and U_e . If $\eta(x, y, t)$ is the absolute invariant of independent variables then

$$g_i(x, y, t, N, U_e, \psi) = F_i(\eta(x, y, t)), \quad i = 1, 2, 3.$$

B. The Invariance Analysis

The transformation of the dependent variables and their partial derivatives are obtained from G via chain-rule operations

$$\left. \begin{aligned} \bar{S}_i &= (C^S / C^i) S_i \\ \bar{S}_{ij} &= (C^S / C^i C^j) S_{ij} \end{aligned} \right\} i, j = x, y, t, \quad (11)$$

where S stands for N, ψ and U_e .

Equation (6) is said to be invariantly transformed under (9) and (11) whenever

$$\begin{aligned} & \bar{\psi}_{\bar{y}\bar{t}} + \bar{\psi}_{\bar{y}} \bar{\psi}_{\bar{y}\bar{x}} - \bar{\psi}_{\bar{x}} \bar{\psi}_{\bar{y}\bar{y}} - \bar{U}_e (\bar{U}_e)_{\bar{x}} \\ & - \nu \left[(1 + \Delta) \bar{\psi}_{\bar{y}\bar{y}\bar{y}} + \Delta \bar{N}_{\bar{y}} + \frac{1}{k_1} (\bar{U}_e - \bar{\psi}_{\bar{y}}) \right] = \quad (12) \\ & H_1(a_1, a_2) \left[\begin{aligned} & \psi_{yt} + \psi_y \psi_{yx} - \psi_x \psi_{yy} - U_e (U_e)_x \\ & - \nu \left[(1 + \Delta) \psi_{yyy} + \Delta N_y + \frac{1}{k_1} (U_e - \psi_y) \right] \end{aligned} \right] \end{aligned}$$

for some function $H_1(a_1, a_2)$ which be constant. Here the transformation in (10) and (11) are for the dependent variables and not derivatives. To transform the partial differential equations, substitution from Eqs. (9) and (11) into the left hand side of (12) and rearrangement yields

$$\begin{aligned} & [c^w / c^y c^t] \psi_{yt} + [(c^w)^2 / (c^y)^2 c^x] \\ & [\psi_y \psi_{yx} - \psi_x \psi_{yy}] - [(c^{U_e})^2 / c^x] [U_e (U_e)_x] \\ & - \nu \left[\begin{aligned} & [c^w / (c^y)^3] (1 + \Delta) \psi_{yyy} + \Delta [c^N / c^y] N_y \\ & + \frac{1}{k_1} (c^{U_e} U_e - [c^w / c^y] \psi_y) \end{aligned} \right] - R \quad (13) \\ & = H_1(a_1, a_2) \left[\begin{aligned} & \psi_{yt} + \psi_y \psi_{yx} - \psi_x \psi_{yy} - U_e (U_e)_x \\ & - \nu \left[\begin{aligned} & (1 + \Delta) \psi_{yyy} + \Delta N_y \\ & + \frac{1}{k_1} (U_e - \psi_y) \end{aligned} \right] \end{aligned} \right] \end{aligned}$$

where

$$R = k^{U_e} \left[\frac{\nu}{k_1} + \left[\frac{c^{U_e}}{c^x} \right] (U_e)_x \right]. \quad (14)$$

For invariant transformation R is equated to zero. This is satisfied by setting

$$R = 0 : k^{U_e} = 0, \quad (15)$$

and comparing the coefficient in both sides of (13) and with $H_1(a_1, a_2)$ we obtain

$$\begin{aligned} & \left[\frac{c^w}{c^y c^t} \right] = \left[\frac{(c^w)^2}{(c^y)^2 c^x} \right] = \left[\frac{c^w}{(c^y)^3} \right] = \left[\frac{c^w}{c^y} \right], \quad (16) \\ & = [c^{U_e}] = \left[\frac{(c^{U_e})^2}{c^x} \right] = \left[\frac{c^N}{c^y} \right] = H_1(a_1, a_2) \end{aligned}$$

where $H_1(a_1, a_2) = \text{constant}$.

In similar manner the invariant transformation of (7) under (9) and (11) whenever there is a function $H_2(a_1, a_2)$ such that

$$\begin{aligned} & [c^N / c^t] N_t + [c^N c^w / c^y c^x] (\psi_y N_x - \psi_x N_y) \\ & + \sigma \left(2 [c^N] N + [c^w / (c^y)^2] \psi_{yy} \right) - \lambda [c^N / (c^y)^2] N_{yy} \\ & + R_1 = H_2(a_1, a_2) \left[\begin{aligned} & N_t + \psi_y N_x - \psi_x N_y + \\ & \sigma (2N + \psi_{yy}) - \lambda N_{yy} \end{aligned} \right], \quad (17) \end{aligned}$$

where

$$R_1 = 2\sigma k^N. \quad (18)$$

The invariance condition implies that

$$\begin{aligned} & [c^N / c^t] = [c^N] = [c^N c^w / c^y c^x] = \\ & [c^w / (c^y)^2] = [c^N / (c^y)^2] = H_2(a_1, a_2), \quad (19) \end{aligned}$$

and

$$R_1 = 0 : k^N = 0. \quad (20)$$

Moreover, the boundary conditions (8) is also invariant in form whenever the condition $k^y=0$ is appended to (15), (16), (19) and (20); that is

$$\bar{y} = 0 : \bar{\psi}_{\bar{y}} = \bar{\psi}_{\bar{x}} = 0, \quad \bar{N} = -\alpha \bar{\psi}_{\bar{y}\bar{y}}, \quad (21)$$

$$\bar{y} \rightarrow \infty : \bar{\psi}_{\bar{y}} \rightarrow \bar{U}_e, \quad \bar{N} \rightarrow 0.$$

Combining equations (15), (16), (19) and (20), we get

$$c^y = c^t = 1 \text{ and } c^w = c^{U_e} = c^N = c^x. \quad (22)$$

Thus, the foregoing restrictions indicate that groups which are of further interest are those in the class C_G with the form

$$G = \left\{ \begin{aligned} & \bar{x} = c^x(a_1, a_2)x + k^x(a_1, a_2) \\ & s : \begin{cases} \bar{y} = y \\ \bar{t} = t + k^t(a_1, a_2) \end{cases} \\ & \bar{\psi} = c^x(a_1, a_2)\psi \\ & \bar{N} = c^x(a_1, a_2)N \\ & \bar{U}_e = c^x(a_1, a_2)U_e, \end{aligned} \right. \quad (23)$$

This group transforms invariantly the differential Eqs. (6) and (7) and the boundary conditions (8).

C. Complete Sets of Absolute Invariants

The complete set of absolute invariants is

- (i) the absolute invariants of the independent variables (x, y, t) is $\eta = \eta(x, y, t)$,
- (ii) the absolute invariants of the dependent variables (ψ, N, U_e) , then

$$g_j(x, y, t; \psi, N, U_e) = F_j(\eta(x, y, t)), \quad j = 1, 2, 3.$$

The basic theorem in group theory (Moran and Gaggioli, 1968) states that: a function $g^*(x, y, t, \psi, N, U_e)$ is an absolute invariant of a two parameter group if it satisfies the following two first-order linear differential equations

$$\sum_{i=1}^{11} (\alpha_i S_i + \alpha_{i+1}) \frac{\partial g^*}{\partial S_i} = 0, \quad (24)$$

$$\sum_{i=1}^{11} (\beta_i S_i + \beta_{i+1}) \frac{\partial g^*}{\partial S_i} = 0, \quad (25)$$

where $S_i = x, y, t, \psi, N, U_e$ and

$$\alpha_1 \equiv \frac{\partial c^x}{\partial a_1}(a_1^0, a_2^0), \alpha_2 \equiv \frac{\partial k^x}{\partial a_1}(a_1^0, a_2^0), \dots, \alpha_{10} \equiv \frac{\partial c^{U_e}}{\partial a_1}(a_1^0, a_2^0), \alpha_{11} \equiv \frac{\partial k^{U_e}}{\partial a_1}(a_1^0, a_2^0), \beta_1 \equiv \frac{\partial c^x}{\partial a_2}(a_1^0, a_2^0), \beta_2 \equiv \frac{\partial k^x}{\partial a_2}(a_1^0, a_2^0), \dots, \beta_{10} \equiv \frac{\partial c^{U_e}}{\partial a_2}(a_1^0, a_2^0), \beta_{11} \equiv \frac{\partial k^{U_e}}{\partial a_2}(a_1^0, a_2^0),$$

where (a_1^0, a_2^0) are the identity elements of the group.

D. Absolute Invariants of Independent Variables

The absolute invariant $\eta = \eta(x, y, t)$ of the independent variables (x, y, t) is determined using Eqs. (24)-(25).

$$\begin{aligned} (\alpha_1 x + \alpha_2) \frac{\partial \eta}{\partial x} + (\alpha_3 y + \alpha_4) \frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6) \frac{\partial \eta}{\partial t} &= 0, \\ (\beta_1 x + \beta_2) \frac{\partial \eta}{\partial x} + (\beta_3 y + \beta_4) \frac{\partial \eta}{\partial y} + (\beta_5 t + \beta_6) \frac{\partial \eta}{\partial t} &= 0. \end{aligned} \tag{26}$$

Since $k^y = 0$, then $\alpha_4 = \beta_4 = 0$, then we have Eq. (25) becomes

$$\begin{aligned} (\alpha_1 x + \alpha_2) \frac{\partial \eta}{\partial x} + (\alpha_3 y) \frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6) \frac{\partial \eta}{\partial t} &= 0, \\ (\beta_1 x + \beta_2) \frac{\partial \eta}{\partial x} + (\beta_3 y) \frac{\partial \eta}{\partial y} + (\beta_5 t + \beta_6) \frac{\partial \eta}{\partial t} &= 0. \end{aligned} \tag{27}$$

The similarity analysis of (6) to (8) now proceeds for the particular case of two parameter groups of the form (23) according to the basic theorem from group theory, Eqs. (27) has only and only one solution if at least one of the following condition is satisfied

$$\left. \begin{aligned} \lambda_{31}x + \lambda_{32} &\neq 0, \\ \lambda_{35}t + \lambda_{36} &\neq 0, \\ \lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26} &= 0. \end{aligned} \right\} \tag{28}$$

where

$$\lambda_{ij} = \alpha_i \beta_j - \alpha_j \beta_i. \quad (i, j=1, 2, 3, 4, 5, 6).$$

For convenience, then, the system (27) will be rewritten in terms of the quantities given by (28); the result is

$$\begin{aligned} (\lambda_{31}x + \lambda_{32}) \frac{\partial \eta}{\partial x} + (\lambda_{35}t + \lambda_{36}) \frac{\partial \eta}{\partial t} &= 0, \\ (\lambda_{31}x + \lambda_{32}) \frac{\partial \eta}{\partial y} - (\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}) \frac{\partial \eta}{\partial t} &= 0. \end{aligned} \tag{29}$$

From the transformation (23) and the definition of α 's, β 's and λ 's, we have the result

$$\lambda_{31} = \lambda_{32} = \lambda_{35} = \lambda_{36} = \lambda_{15} = \lambda_{25} = 0,$$

which implies

$$\left. \begin{aligned} \lambda_{31}x + \lambda_{32} &= 0 \\ \lambda_{35}t + \lambda_{36} &= 0 \end{aligned} \right\} \tag{30}$$

then, the conditions (28) reduce to

$$\lambda_{16}x + \lambda_{26} \neq 0. \tag{31}$$

Applying Eqs. (30) and (31) to Eqs. (29) gives

- (i) the first equation of (29) is identically satisfied,
- (ii) the second equation of (29) reduce to

$$\frac{\partial \eta}{\partial t} = 0. \tag{32}$$

For convenience, Eqs. (29) can be rewritten in the form

$$(\lambda_{16}x + \lambda_{26}) \frac{\partial \eta}{\partial x} = 0, \tag{33}$$

and from (31), the Eq. (33) gives

$$\frac{\partial \eta}{\partial x} = 0. \tag{34}$$

From Eqs. (32) and (34) we have

$$\eta = f(y). \tag{35}$$

Without loss of generality the independent absolute invariant $\eta(y)$ in Eq. (35) may assume of the form

$$\eta = Ay. \tag{36}$$

E. Absolute Invariants of Dependent Variables

For the absolute invariants corresponding to the dependent variable ψ, N and U_e . A function $g_1(x, t, \psi)$ is absolute invariant of a two-parameter group if it satisfies the first-order linear differential equations

$$\begin{aligned} (\alpha_1 x + \alpha_2) \frac{\partial g_1}{\partial x} + (\alpha_3 t + \alpha_4) \frac{\partial g_1}{\partial t} + (\alpha_5 \psi + \alpha_6) \frac{\partial g_1}{\partial \psi} &= 0, \\ (\beta_1 x + \beta_2) \frac{\partial g_1}{\partial x} + (\beta_3 t + \alpha_4) \frac{\partial g_1}{\partial t} + (\beta_5 \psi + \beta_6) \frac{\partial g_1}{\partial \psi} &= 0. \end{aligned} \tag{37}$$

The solution of Eqs. (37) gives

$$g_1(x, t, \psi) = \phi_1(\psi / \omega_1(x, t)) = F(\eta). \tag{38}$$

In similar manner, we get

$$g_2(x, t, N) = \phi_2(N / \omega_2(x, t)) = G(\eta), \tag{39}$$

$$g_3(x, t, U_e) = \phi_3(U_e / \omega_3(x, t)) = E(\eta). \tag{40}$$

where $\omega_1(x, t), \omega_2(x, t)$ and $\omega_3(x, t)$ are functions to be determined. Without loss of generality, the ϕ 's in (38), (39) and (40) are selected to be the identity functions. Then we can express the functions $\psi(x, y, t), N(x, y, t)$ and $U_e(x, y, t)$ in terms of the absolute invariants $F(\eta)$ and $G(\eta)$ in the form

$$\psi(x, y, t) = \omega_1(x, t)F(\eta), \tag{41}$$

$$N(x, y, t) = \omega_2(x, t)G(\eta), \tag{42}$$

$$U_e(x, t) = \omega_3(x, t)E(\eta). \tag{43}$$

Since ω_3 is independent of y , whereas η depends on y , it follow that E in (43) must be equal to a constant. Then (43) becomes

$$U_e(x, t) = U_0 \omega_3(x, t). \tag{44}$$

We follow the work of many recent authors by assuming that U_e is given by (Lok *et al.*, 2003a)

$$U_e(x, t) = U_e(x) = ax. \tag{45}$$

The forms of the functions ω_1 and ω_2 are those for which the governing Eqs. (6) and (8) reduce to ordinary differential equations.

IV. THE REDUCTION TO ORDINARY DEFFERENTIAL EQUATION

Here, we will consider two cases

A Steady Case (i.e. $\omega_1(x,t) = \omega_1(x)$ and $\omega_2(x,t) = \omega_2(x)$)

As a special case of our study, we can take $\omega_1(x)$, $\omega_2(x)$ and A in the form

$$\begin{cases} \omega_1(x) = ax\sqrt{v/a} \\ \omega_2(x) = ax\sqrt{a/v} \\ A = \sqrt{a/v}, \end{cases} \quad (46)$$

then, η , ψ and N become

$$\begin{cases} \eta = \sqrt{a/v}y \\ \psi = ax\sqrt{v/a}F(\eta) \\ N = ax\sqrt{a/v}G(\eta), \end{cases} \quad (47)$$

then, Eqs. (6) and (7) give

$$(1+\Delta)F'''+\Delta G'+[FF'-F'^2]+\frac{1}{m}(1-F')+1=0, \quad (48)$$

$$\lambda G''+[FG'-F'G]-\sigma(2G+F'')=0, \quad (49)$$

with the boundary conditions

$$F(0) = 0, F'(0) = 0, G(0) = -\beta F''(0), \quad (50)$$

$$F \rightarrow 1, G \rightarrow 0 \text{ as } \eta \rightarrow \infty,$$

where $m=v/k_1a$ is permeability parameter.

The most important results are the local wall shear stress τ_w and the local wall couple stress M_w which may be written as

$$\tau_w = \left[[\mu+k] \frac{\partial u}{\partial y} + kN \right]_{y=0} = \sqrt{\frac{a^3}{v}} x \mu [1 + [1-\beta]\Delta] F''(0), \quad (51)$$

$$M_w = \gamma \left[\frac{\partial N}{\partial y} \right]_{y=0} = \frac{\gamma a^2 x}{v} G'(0), \quad (52)$$

B. Unsteady Case

In this case, also, as a special case of our work and according to Eq. (34), we will introduce the (Lok *et al.*, 2003b) transformations for the dimensionless stream function F , the dimensionless microrotation function G and a pseudo-similarity η which are given by

$$\begin{cases} \psi = 2\sqrt{vt}axF(\eta,t) \\ N = \frac{ax}{2\sqrt{vt}}G(\eta,t) \\ \eta = \frac{y}{2\sqrt{vt}} \\ \tau = 2\sqrt{at}, \end{cases} \quad (53)$$

The outcome of this transformation is that the Eqs. (6) and (7) become

$$(1+\Delta)\frac{\partial^3 F}{\partial \eta^3} + \Delta \frac{\partial G}{\partial \eta} + 2\eta \frac{\partial^2 F}{\partial \eta^2} - 2\tau \frac{\partial^2 F}{\partial \eta \partial \tau} + \tau^2 \left[1 + \frac{1}{m} \left(1 - \frac{\partial F}{\partial \eta} \right) - \left(\frac{\partial F}{\partial \eta} \right) + F \frac{\partial^2 F}{\partial \eta^2} \right] = 0, \quad (54)$$

$$\lambda \frac{\partial^2 G}{\partial \eta^2} + 2\eta \frac{\partial G}{\partial \eta} + 2G - 2\tau \frac{\partial G}{\partial \tau} \quad (55)$$

$$\tau^2 \left[F \frac{\partial G}{\partial \eta} - G \frac{\partial F}{\partial \eta} - \sigma \left(2G + \frac{\partial^2 F}{\partial \eta^2} \right) \right] = 0,$$

and the boundary conditions (8) transform to

$$F = \frac{\partial F}{\partial \eta} = 0, G = -\beta \frac{\partial^2 F}{\partial \eta^2} \text{ on } \eta = 0, \quad (56)$$

$$\frac{\partial F}{\partial \eta} \rightarrow 1, G \rightarrow 0 \text{ as } \eta \rightarrow \infty.$$

The initial velocity and microrotation profiles $F(\eta)$ and $G(\eta)$ at $t=0$ are obtained from the following ordinary differential equations

$$(1+\Delta)F'''+\Delta G'+2\eta F''=0, \quad (57)$$

$$\lambda G''+2\eta G'+2G=0, \quad (58)$$

Subject to the boundary conditions

$$F(0) = F'(0) = 0, G(0) = -\beta F''(0), \quad (59)$$

$$F' \rightarrow 1, G \rightarrow 0 \text{ as } \eta \rightarrow \infty,$$

where the primes denote differentiation with respect to η . The analytical solutions of these equations are

$$F'(\eta) = \text{erf} \left[\frac{\eta}{(1+\Delta)^{\frac{1}{2}}} \right] + 2\beta \left[\frac{\lambda}{(1+\Delta)} \right]^{\frac{1}{2}} \times \left[1 - 2\beta + 2\beta \left[\frac{\lambda}{(1+\Delta)} \right]^{\frac{1}{2}} \right]^{-1} \quad (60)$$

$$\times \left\{ \text{erf} \left[\frac{\eta}{(1+\Delta)^{\frac{1}{2}}} \right] + \text{erf} \left[\frac{\eta}{\sqrt{\lambda}} \right] \right\},$$

$$G(\eta) = \frac{-2\beta}{\sqrt{\pi(1+\Delta)}} \left[1 - 2\beta + 2\beta \left[\frac{\lambda}{1+\Delta} \right]^{\frac{1}{2}} \right]^{-1} \quad (61)$$

$$\times \exp \left[-\frac{\eta^2}{\lambda} \right].$$

Also, we notice that for $n=1/2$ (weak concentration of microelement), we can take

$$G(\eta) = -\frac{1}{2} \frac{\partial^2 F}{\partial \eta^2}, \quad (62)$$

and the velocity profiles can be described by the following equations

$$\left(1 + \frac{\Delta}{2} \right) \frac{\partial^3 F}{\partial \eta^3} + 2\eta \frac{\partial^2 F}{\partial \eta^2} - 2\tau \frac{\partial^2 F}{\partial \eta \partial \tau} + \tau^2 \left[1 + \frac{1}{m} \left(1 - \frac{\partial F}{\partial \eta} \right) - \left(\frac{\partial F}{\partial \eta} \right) + F \frac{\partial^2 F}{\partial \eta^2} \right] = 0. \quad (63)$$

$$\tau^2 \left[1 + \frac{1}{m} \left(1 - \frac{\partial F}{\partial \eta} \right) - \left(\frac{\partial F}{\partial \eta} \right) + F \frac{\partial^2 F}{\partial \eta^2} \right] = 0.$$

Subject to the boundary conditions

$$F = \frac{\partial F}{\partial \eta} = 0, \text{ on } \eta = 0 \tag{64}$$

$$\frac{\partial F}{\partial \eta} \rightarrow 1, \text{ as } \eta \rightarrow \infty.$$

The remaining results in this case are the local wall shear stress τ_w , and local wall couple stress M_w which may be written as

$$\tau_w = \left[[\mu + k] \frac{\partial u}{\partial y} + kN \right]_{y=0} \tag{65}$$

$$= \frac{ax\mu}{2\sqrt{vt}} [1 + [1 - \beta]\Delta] F'''(0),$$

$$M_w = \gamma \left[\frac{\partial N}{\partial y} \right]_{y=0} = \frac{\gamma ax}{4\sqrt{vt}} G'(0). \tag{66}$$

V. RESULTS AND DISCUSSION

A two parameter transformation group was applied to a system of equations governing the unsteady boundary layer flow of a micropolar fluid near the rear stagnation point of a plane surface in a porous medium. The similarity representations and similarity transformations in the steady case was described by Eqs. (48) and (49). These equations with the boundary conditions (50) were solved numerically using a fourth-order Rung-Kutta method. Systematic "shooting" is required to satisfy their boundary conditions at infinity. Calculations were carried out for the indicated values of the micropolar parameter Δ and the permeability parameter m are summarized with $\lambda=2.0$ and $\sigma=0.5$. The step size $\Delta\eta=0.05$ is while obtaining the numerical solution with $\eta_\infty=4$ and five-decimal accuracy as the criterion convergence. In order to assess the accuracy of the present results, it was found in a full agreement with the work of Lok *et al.* (2003b)".

The effects of vortex-viscosity parameter Δ and permeability parameter m on the variation of velocity for $\beta=0$ (strong concentration of microelement) are shown in Fig. 2 and 3. It is found that, the velocity distributions decrease with an increasing the values of micropolar parameter Δ and permeability parameter m and

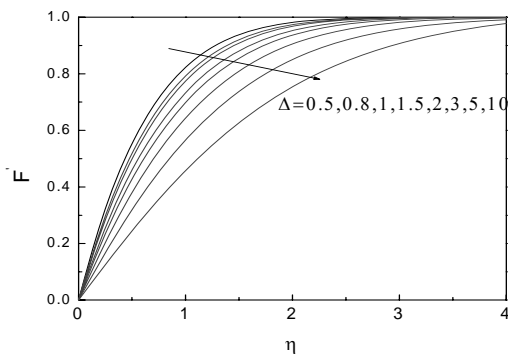


Fig. 2. Variation of velocity distribution with vortex-viscosity parameter when $\beta=0$ (strong concentration of microelement)

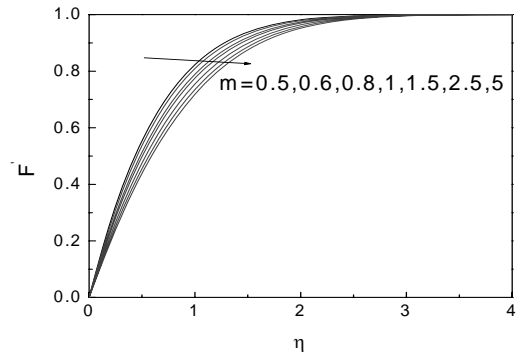


Fig. 3. Variation of velocity distribution with permeability parameter when $\beta=0$ (strong concentration of microelement)

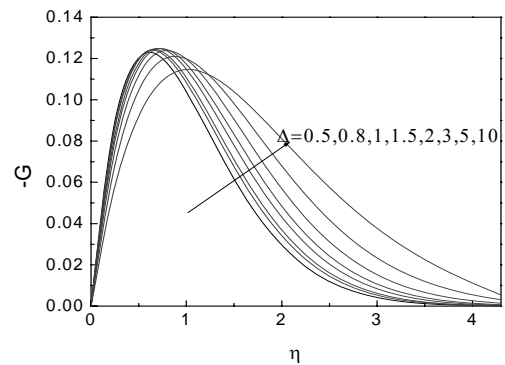


Fig. 4. Variation of angular velocity with vortex-viscosity parameter when $\beta=0$ (strong concentration of microelement)

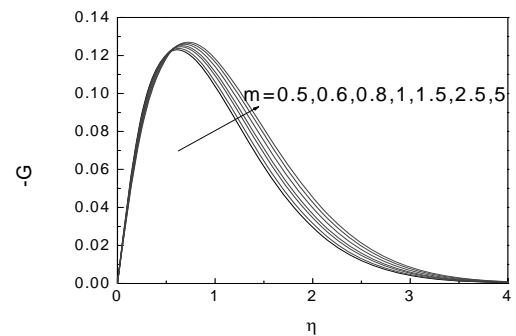


Fig. 5. Variation of angular velocity with permeability parameter when $\beta=0$ (strong concentration of microelement)

the velocity distributions assumes a more uniform shape within the boundary layer. Figures 4 and 5 depicts the effects of material parameter Δ and permeability parameter m on the variation of microrotation profiles for $\beta=0$ (strong concentration of microelement). From these figures, it can be seen that, as the values of Δ and m increase the rotation of microelement increases until reach a maximum and then decrease to zero.

The variations of velocity and microrotation profiles with the material parameter Δ for $\beta=1/2$ (weak concentration of microelement) are shown in figures 6 and 7. It is observed from these figures that, for the same values

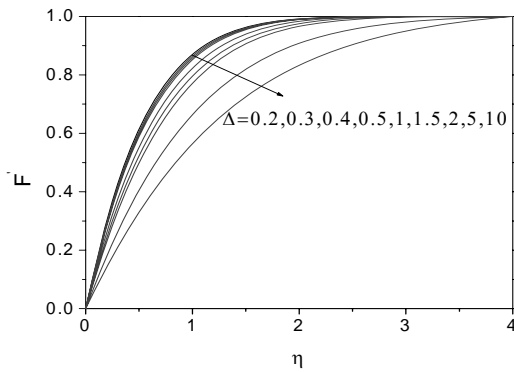


Fig. 6. Variation of velocity distribution with vortex-viscosity when $\beta=0.5$ (weak concentration of microelement)

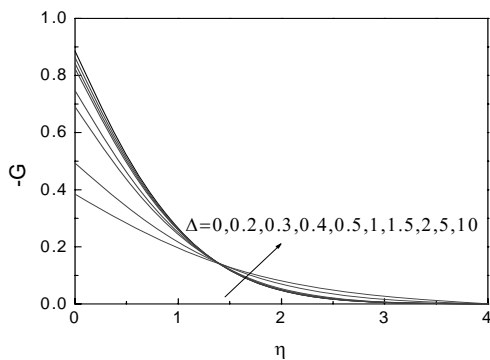


Fig. 7. Variation of angular velocity with vortex-viscosity when $\beta=0.5$ (weak concentration of microelement)

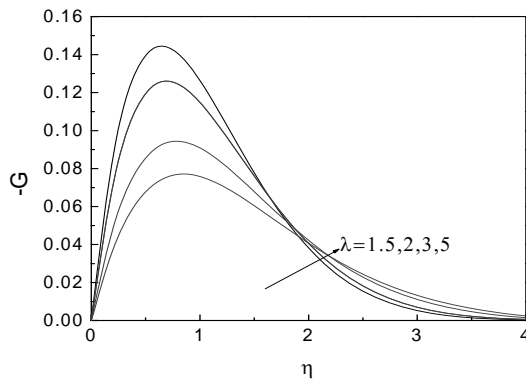


Fig. 8. variation of angular velocity with microrotation parameter

of Δ the velocity profiles are higher for $\beta=0.5$ than for $\beta=0$ and the microrotation profiles decrease from maximum values at the wall to zero. A figure 8 displays the effect of microrotation parameter λ on the variation of angular velocity profiles. It is shown that, as λ increase the angular velocity profiles decrease beside the wall and increase far from the wall.

The comparing between our work and the previous work for $\beta=0$ which reported by Lok *et al.* (2003b) is described in Fig. 9 and 10. We observed from these fig-

ures that, the obtained results agree very well with the previous studies for the same values of Δ . In addition, for the different values of the micro-rotation boundary condition β , the effects of variations in flow conditions and fluid properties on the variations of $F'''(0)$ and $G'(0)$ at the plate are illustrated in table 1. The referenced case is $\lambda=2.0$ and $\sigma=0.5$. From this table, the results show that, increasing values of vortex-viscosity parameter Δ and permeability parameter m results in a decrease of $F'''(0)$ whereas increasing values of Δ and m results in an increasing of $G'(0)$.

For the unsteady case the similarity representations are found in system (54) and (55) with the boundary conditions (56). The analytical solutions form for the initial velocity in this case was formulated by the integration of equations (57) and (58) with the boundary conditions (59). The most important results in this case are the local wall shear stress τ_w and the local wall couple stress M_w which may be written in Eqs. (65) and (66).

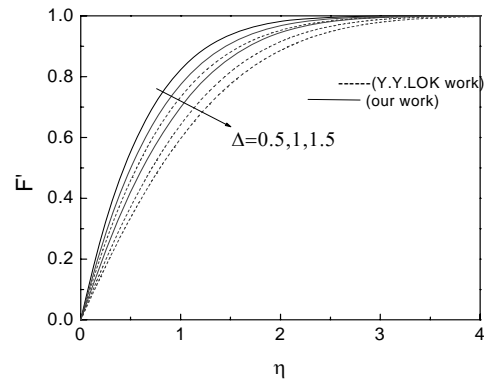


Fig. 9. Variation of velocity distribution with vortex-viscosity when $\beta=0$

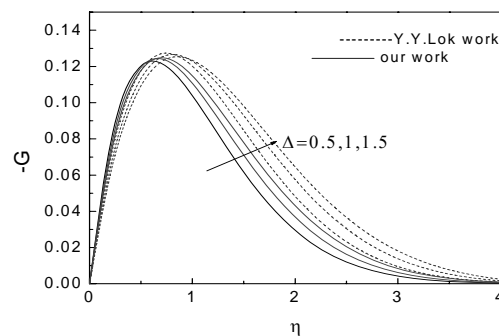


Fig. 10. variation of angular velocity with vortex-viscosity when $\beta=0$

VI. CONCLUSIONS

A solution methodology based on the group theoretic method has been applied to solve the problem of unsteady boundary layer flow of a micropolar fluid near the rear stagnation point of a plane surface in a porous medium. The carried out analysis, in this problem,

shows the effectiveness of the method in obtaining invariant solutions for the system of partial differential equations. The main difficulty of the problem is due the nonlinear boundary conditions, and consequently the analysis of the problem faces many difficulties that were an essential obstacle in many other analytical methods.

In this problem, we have used via-chain rule under translation group of transformations to reduce the number of independent variables of the problem by two and consequently the governing partial differential equations with the boundary conditions to ordinary differential equations with the appropriate corresponding boundary conditions. Numerical technique based on the shooting method was used. The obtained results were presented and discussed. It is found that, as the values of material parameter Δ increase, the velocity distribution becomes more linear and as the permeability parameter increase, the velocity distribution assumes a more uniform shape within the boundary layer. In addition, it was found that as the values of Δ and m increase the rotation of microelement increases until reach a maximum and then decrease to zero. The form of the local wall shear stress τ_w and the local wall couple stress M_w in steady and unsteady cases was derived. The analytical form for the initial velocity in unsteady case was formulated. Comparisons with previously published work on various special cases of the general problem were performed and the results were found to be in excellent agreement.

Table 1. Variation of $F''(0), -G'(0)$ at the plate with m, Δ :

		At $\beta=0$, (strong concentration of microelement)		At $\beta=0.5$, (weak concentration of microelement)	
m	Δ	$F''(0)$	$-G'(0)$	$F''(0)$	$G'(0)$
0.5	0.5	1.50237	0.51645	1.64712	0.70334
0.5	0.8	1.35988	0.49407	1.54924	0.64098
0.5	1	1.28366	0.48115	1.49398	0.60646
0.5	1.5	1.13582	0.45391	1.38036	0.53715
0.5	1.8	1.06730	0.44018	1.32455	0.50398
0.5	2	1.02760	0.43187	1.29123	0.48446
0.5	2.5	0.94411	0.41345	1.21862	0.44270
0.5	2.8	0.90236	0.40373	1.18095	0.42147
0.5	3	0.87726	0.39770	1.15785	0.40860
0.5	3.5	0.82223	0.38401	1.10592	0.38011
0.5	4	0.77595	0.37193	1.06084	0.35587
0.5	4.5	0.73636	0.36117	1.02118	0.33495
0.5	5	0.70201	0.35149	0.98593	0.31667
0.5	0.5	1.50237	0.51645	1.56960	0.65357
0.6	0.5	1.42802	0.50571	1.46660	0.58840
0.8	0.5	1.32946	0.49073	1.40097	0.54748
1	0.5	1.26680	0.48072	1.30806	0.49042
1.5	0.5	1.17829	0.46590	1.27550	0.47068
1.8	0.5	1.14734	0.46052	1.25889	0.46066
2	0.5	1.13156	0.45773	1.24513	0.45238
2.2	0.5	1.11850	0.45540	1.22841	0.44236
2.5	0.5	1.10262	0.45255	1.20763	0.42997

NOMENCLATURE

- F reduced stream function
- G reduced microrotation
- j micro inertia density [m^2]
- α ratio of the microrotation vector component
- N component of the microrotation vector normal to x - y plan [s^{-1}]
- t time [s]
- u, v velocity components along x and y axes [ms^{-1}]
- U_e uniform stream velocity [ms^{-1}]
- x, y Cartesian coordinates along the wall and normal to it [m]

Greek symbols

- γ spin gradient viscosity [$kg.m.s^{-1}$]
- η pseudo-similarity variables
- k vortex-viscosity [$kg.m.s^{-1}$]
- μ viscosity [$kg.m.s^{-1}$]
- ν kinematic viscosity [$m^2.s^{-1}$]
- ρ density [$kg.m^3$]
- ψ stream function

Subscripts

- e boundary layer edge condition
- ∞ far field condition

Superscript

- $'$ differentiation with respect to η

REFERENCES

Abd-el-Malek, M.B., M. Kassem and M.L. Makky, "similarity solutions for unsteady free convection flow from a continuous moving vertical surface", *J. Comput. Appl. Math.*, **164**, 11-24 (2004).

Eringen, A., "Simple microfluids", *Int. J. Eng. Sci.*, **2**, 205-217 (1964).

Eringen, A., "Theory of micropolar fluids", *J. Math. Mech.*, **16**, 1-18 (1966).

Eringen, A., "Theory of thermomicropolar fluids", *J. Math. Anal. Appl.*, **9**, 480-496 (1972).

Helal, M.M. and M.B. Abd-el-Malek, "Group method analysis of magneto-elastico-viscous flow along a semi-infinite flat plat with heat transfer", *J. Comput. Appl. Math.*, **173**, 199- 210 (2005).

Lok, Y.Y., N. Amin, I. Pop and P. Phang, "Unsteady boundary layer flow of a micropolar flow near the forward stagnation point of a plane surface", *Int. J. Eng. Sci.*, **4**, 173-186 (2003a).

Lok, Y.Y., N. Amin and I. Pop, "Unsteady boundary layer flow of a micropolar flow near the rear stagnation point of a plane surface", *Int. J. Ther. Sci.*, **42**, 995-1001 (2003b).

Moran, M.J. and R.A. Gaggioli, "Reduction of the number of variables in systems of partial differential equations with auxiliary conditions", *SIAM. J. Appl. Math.*, **16**, 202-215 (1968).

Seshadri, R., N. Sreeshylan and G. Nath, "Unsteady mixed convection flow in the tagnation region of a heated vertical plate due to impulsive motion", *Int. J. Heat Mass Transfer*, **45**, 1345-1352 (2002).

Received: August 2, 2007.

Accepted: August 20, 2007.

Recommended by Subject Editor: Eduardo Dvorkin