UNIVERSAL CONSTRUCTION OF CONTROL LYAPUNOV FUNCTIONS FOR LINEAR SYSTEMS *

X. S. CAI[†,‡] and Z. Z. HAN[†]
† Department of Automation, Shanghai Jiaotong University, 200030, Shanghai, China
‡ Department of Mathematics, Sanming University, 365001, Fujian, China
xiushan@zjnu.cn
zzhan@sjtu.edu.cn

Abstract– This paper develops a method by which control Lyapunov functions of linear systems can be constructed systematically. It proves that the method can provide all quadratic control Lyapunov functions for a given linear system. By using the control Lyapunov function a linear feedback is established to stabilize the linear system. Moreover, it can also assign poles of the closed-loop system in the position designed in advance.

Keywords– linear systems, stabilization, control Lyapunov functions

I. INTRODUCTION

In the early days of control theory investigation, most of concepts such as stability, optimality and uncertainty were descriptive rather than constructive. The situation has been gradually changed in the last two decades. Kokotovic and Arcak (2001) made a survey for the alteration and call it 'activation'. A prominent example of the activation is the concept of control Lyapunov function (henceforth CLF for short). Traditionally, Lyapunov function is a powerful tool to the analysis of stability of dynamic systems. Artstein (1983) and Sontag (1983) considered respectively the stabilization of control systems and extended the notion of Lyapunov function to that of control Lyapunov function. It has been verified that a nonlinear system can be stabilized by a relaxed state feedback if and only if it holds a CLF (Artstein 1983). Moreover, Sontag (1989) dealt with the stabilization of affine systems and presented a universal feedback scheme by using CLF. These achievements greatly motivated the investigation of CLF, and CLF were widely adopted in various design problems. For instance, from Freeman and Primbs (1996), Freeman and Kokotovic (1996) Liberzon *et al.* (2002), Cai and Han (2005), Sepulchre *et al.*, (1997), the readers can find many meaningful results.

However, being similar to the situation of Lyapunov function, a CLF is not always available for a given system, even for a linear system. We have no a general method to construct a CLF. Hence, the construction of CLF becomes the bottleneck of the design technique developed by using CLF.

This paper presents a systematic study for CLF of linear systems. We give the necessary and sufficient conditions for CLF of linear systems. We establish a method to construct a quadratic CLF for a linear system by solving a Lyapunov equation. Freeman and Primbs (1996) also gave an approach to obtain a CLF for a linear system by solving a Riccati equation. It is clear that Lyapunov equation is much simpler than Riccati equation since the former is linear and the later is quadratic. We then prove that for a linear system there exists a quadratic CLF if it has a CLF. A linear feedback by CLF is designed to stabilize the given system.

The significance of the CLF comes from the universal formulas. After the works of Sontag (1989), there are a number of universal feedback schemes presented for the stabilization, tracing, regulation, robust control, optimization and so on. A toolbox for the design using CLF is then easily developed. It means that compensators for the design problems mentioned above can be achieved if a CLF is available. These design techniques can be applied to a linear system if a CLF is obtained although it will lead to a nonlinear system. Another significance of the investigation of CLF of linear systems is that it may open a way to the construction of CLF for affine systems by the zero dynamic method (Isidori 1989) and for the general nonlinear systems by the central

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manifold method.

The organization of the paper is as follows: Section 2 provides the preliminaries of the paper that include the description of the problem, the definition of CLF and some fundamental results. Section 3 gives the main results of the paper. We present a method to construct CLFs for a linear system and verify that every quadratic CLF can be obtained by the method. A linear feedback is proposed based on the CLF to stabilize the linear system. Section 4 summarizes the CLF of uncontrollable linear system. Section 5 includes the algorithm for constructing CLFs. The last section concludes the paper.

II. PRELIMINARIES

Consider the linear time-invariant system described by

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the input.

Let $V : R^n \to R$ be a differentiable function. V is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$; V is said to be proper if $V(x) \to \infty$ as $\|x\| \to \infty$.

Definition (Artstein, 1983) If there exists a differentiable, proper and positive definite function V such that

$$\inf_u \frac{\partial V}{\partial x}(Ax + Bu) < 0 \tag{2}$$

for each $x \neq 0$, then $V(x)$ is a control Lyapunov function (CLF) for the system (1).

If u is a determined function, the notation of $\inf\limits_u$ can be drawn away, then (2) is exactly the requirement of a Lyapunov function which is used to determine the stability of the system. But in the linear control system u is undetermined, we have to add the infimum before the inequality. It is clear that (2) is equivalent to the following statement

$$\frac{\partial V}{\partial x}B = 0, x \neq 0 \Rightarrow \frac{\partial V}{\partial x}Ax < 0. \tag{3}$$

To end this section, we give three Propositions that describe the invariance of CLF.

Proposition 1 Let T be a nonsingular real matrix. By a coordinate transformation $\bar{x} = Tx$, the system (1) becomes

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu. \tag{4}$$

Then $V(x)$ is a CLF for the system (1) if and only if $V_1(\bar{x}) = V(T^{-1}\bar{x})$ is a CLF for the system (4). $\diamond$

Proposition 2 If $G \in R^{m\times m}$ is invertible, then $V(x)$ is a CLF for the system (1) if and only if it is a CLF for the system $\dot{x} = Ax + BGv$. $\diamond$

Proposition 3 For $F \in R^{m\times n}$, the feedback takes the form of $u = Fx + v$. $V(x)$ is a CLF for the system $\dot{x} = (A + BF)x + Bv$ if and only if it is a CLF for the system (1). $\diamond$

Proofs of these Propositions are straightforward, and hence omitted.

III. CLFS FOR CONTROLLABLE SYSTEMS

This section presents a method of construction of CLF for linear systems. We start with the construction of CLFs for the single-input system and then extend results to the multi-input system. A linear feedback is also obtained by the CLF to stabilize the system.

A. CLFs for A Single-Input Controllable System

We consider the case of $m = 1$ in this subsection. If (1) is controllable, without loss of the generality, we assume that (A, B) takes its Brunovsky canonical form. *i.e.* in the system (1),

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

Divide A and x into their block form $A = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & -\alpha_{n-1} \end{bmatrix}$, where

$$G_{11} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, G_{12} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$G_{21} = \begin{bmatrix} -\alpha_0 & \cdots & -\alpha_{n-2} \end{bmatrix},$$

and $x^T = \begin{bmatrix} X_{n-1}^T & x_n \end{bmatrix}, X_{n-1}^T = \begin{bmatrix} x_1 & \cdots & x_{n-1} \end{bmatrix}$.

After the state feedback $u = -\begin{bmatrix} G_{21} & -\alpha_{n-1} \end{bmatrix} x + \bar{u}$, the system becomes

$$\dot{x} = A_c x + B\bar{u} \tag{6}$$

where $A_c = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}$. By Proposition 3, it is sufficient to study CLFs for the system (6).

This section considers quadratic CLFs of the form $V(x) = x^T P x$, where P is a symmetric matrix. Divide P into a block form as follows

$$P = \begin{bmatrix} P_{n-1} & P_{12} \\ P_{12}^T & p_{22} \end{bmatrix}$$

where $P_{n-1} \in R^{(n-1)\times(n-1)}$, $p_{22} \in R$ and $P_{12} \in R^{n-1}$. Denote $p_{22}^{-1}P_{12}^T = [\ \beta_1 \ \cdots \ \beta_{n-1}\ ]$.

Let C_β be the companion matrix of $p_{22}^{-1}P_{12}^T$. Then

$$C_\beta = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\beta_1 & -\beta_2 & \cdots & -\beta_{n-1} \end{bmatrix}. \quad (7)$$

The characteristic polynomial of C_β is

$$\lambda(\beta) = \lambda^{n-1} + \beta_{n-1}\lambda^{n-2} + \cdots + \beta_2\lambda + \beta_1.$$

The following Conditions are proposed for the further discussion.

H1 $p_{22} > 0$ and $\lambda(\beta)$ is a Hurwitz polynomial.

H2 $(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)C_\beta + C_\beta^T(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)$ is negative definite.

There are three Remarks to the above Conditions.

Remark 1 $\lambda(\beta)$ is a Hurwitz polynomial if and only if C_β is a Hurwitz matrix.

Remark 2 H2 implies $P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T$ is a positive definite matrix provided that C_β is a Hurwitz matrix.

Remark 3 Because

$$\begin{bmatrix} I & -p_{22}^{-1}P_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{n-1} & P_{12} \\ P_{12}^T & p_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -p_{22}^{-1}P_{12}^T & 1 \end{bmatrix} = \begin{bmatrix} P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T & 0 \\ 0 & p_{22} \end{bmatrix}$$

H1 and H2 imply that P is a positive definite matrix.

The following subsection will verify that $V(x)$ is a CLF for the system (6) if and only if $V(x)$ satisfies Conditions H1 and H2.

Construction of a CLF

This subsection considers the system (6).

Theorem 1 $V(x) = x^TPx$ is a CLF for the system (6) if and only if P satisfies Conditions H1 and H2.

Proof: (Sufficiency) By Remark 3, if Conditions H1 and H2 hold, this P is positive definite. The derivative of $V(x) = x^TPx$ along the system (6) is

$$\begin{aligned}\dot V(x) &= x^TPA_cx + x^TA_c^TPx + 2x^TPB\bar u \\ &= [\ X_{n-1}^T \ \ x_n\ ] \times \\ &\begin{bmatrix} P_{n-1}G_{11} + G_{11}^TP_{n-1} & P_{n-1}G_{12} + G_{11}^TP_{12} \\ P_{12}^TG_{11} + G_{12}^TP_{n-1} & P_{12}^TG_{12} + G_{12}^TP_{12} \end{bmatrix} \times \\ &\begin{bmatrix} X_{n-1} \\ x_n \end{bmatrix} + (2X_{n-1}^TP_{12} + 2x_np_{22})\bar u.\end{aligned} \quad (8)$$

From (8), we obtain

$$\tfrac{\partial V}{\partial x}B = 2X_{n-1}^TP_{12} + 2x_np_{22},$$

and

$$\begin{aligned}&\tfrac{\partial V}{\partial x}A_cx = [\ X_{n-1}^T \ \ x_n\ ] \times \\ &\begin{bmatrix} P_{n-1}G_{11} + G_{11}^TP_{n-1} & P_{n-1}G_{12} + G_{11}^TP_{12} \\ P_{12}^TG_{11} + G_{12}^TP_{n-1} & P_{12}^TG_{12} + G_{12}^TP_{12} \end{bmatrix} \times \\ &\begin{bmatrix} X_{n-1} \\ x_n \end{bmatrix}.\end{aligned} \quad (9)$$

Then $\frac{\partial V}{\partial x}B = 0$ implies

$$x_n = -X_{n-1}^Tp_{22}^{-1}P_{12}. \quad (10)$$

Substituting (10) to (9), we obtain

$$\begin{aligned}&\tfrac{\partial V}{\partial x}A_cx \\ &= X_{n-1}^T[(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)(G_{11} - G_{12}p_{22}^{-1}P_{12}^T) \\ &+(G_{11} - G_{12}p_{22}^{-1}P_{12}^T)^T(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)]X_{n-1}.\end{aligned} \quad (11)$$

Since

$$G_{11} - G_{12}p_{22}^{-1}P_{12}^T = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\beta_1 & -\beta_2 & \cdots & -\beta_{n-1} \end{bmatrix} = C_\beta,$$

by H2, $\frac{\partial V}{\partial x}A_cx < 0$ for $X_{n-1} \neq 0$.
Thus $\frac{\partial V}{\partial x}B = 0, x \neq 0$, implies $\frac{\partial V}{\partial x}A_cx < 0$.
$V(x) = x^TPx$ is indeed a CLF of the system (6).

(Necessity) P is positive definite, hence $p_{22} > 0$. From (8), $\frac{\partial V}{\partial x}B = 0$ implies $x_n = -p_{22}^{-1}X_{n-1}^TP_{12}$, and from (8) again

$$\begin{aligned}\tfrac{\partial V}{\partial x}A_cx &= X_{n-1}^T[(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)(G_{11} - G_{12}p_{22}^{-1}P_{12}^T) \\ &+(G_{11} - G_{12}p_{22}^{-1}P_{12}^T)^T(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)]X_{n-1}.\end{aligned}$$

Denote

$$\begin{aligned}&(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T)(G_{11} - G_{12}p_{22}^{-1}P_{12}^T) \\ &+(G_{11} - G_{12}p_{22}^{-1}P_{12}^T)^T(P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T) = -Q.\end{aligned} \quad (12)$$

From the definition of CLF,

$$\frac{\partial V}{\partial x}B = 0, x \neq 0 \Rightarrow \frac{\partial V}{\partial x}A_cx < 0. \quad (13)$$

Then Q is positive definite. Thus H2 is satisfied.

On the other hand $P = \begin{bmatrix} P_{n-1} & P_{12} \\ P_{12}^T & p_{22} \end{bmatrix}$ is positive definite, and by the Remark 3, $P_{n-1} - p_{22}^{-1}P_{12}P_{12}^T$ is positive definite too. Hence the Lyapunov Theorem shows that $G_{11} - G_{12}p_{22}^{-1}P_{12}^T$ is stable, *i.e.*

$$G_{11} - G_{12}p_{22}^{-1}P_{12}^T = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\beta_1 & -\beta_2 & \cdots & -\beta_{n-1} \end{bmatrix}$$

is a Hurwitz matrix. H1 is also satisfied. ⋄

We emphasize that $V(x) = x^T Px$ is also a CLF for the system (5) by Proposition 3.

Example 1 illustrates the constructing method of CLFs presented by Theorem 1.

Example 1 Consider the linear system (5) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Assume $P = \begin{bmatrix} P_3 & P_{12} \\ P_{12}^T & p_{22} \end{bmatrix}$, where $p_{22}^{-1} P_{12}^T = \begin{bmatrix} 6 & 11 & 6 \end{bmatrix}$, $p_{22} = 4$.
To obtain P_3, we consider the following Lyapunov equation:

$$(P_3 - p_{22}^{-1} P_{12} P_{12}^T) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}^T (P_3 - p_{22}^{-1} P_{12} P_{12}^T) = -I_3. \tag{14}$$

Solving equation (14), we have

$$P_3 = \begin{bmatrix} 145.8167 & 265.15 & 144.0833 \\ 265.15 & 486.0083 & 264.15 \\ 144.0833 & 264.15 & 144.1083 \end{bmatrix}.$$

By Theorem 1,

$$P = \begin{bmatrix} 145.8167 & 265.15 & 144.0833 & 24 \\ 265.15 & 486.0083 & 264.15 & 44 \\ 144.0833 & 264.15 & 144.1083 & 24 \\ 24 & 44 & 24 & 4 \end{bmatrix}$$

is a positive definite matrix, and
$V(x) = x^T Px$ is a CLF for this system.

Remark 4 Freeman and Primbs (1996) also offered a method to construct CLFs for the linear system (1). They showed that the positive definite solution P of the Riccati equation $A^T P + PA + Q - PBR^{-1}B^T P = 0$ provides a quadratic CLF $V(x) = x^T Px$, where Q and R are positive semidefinite and positive definite respectively. The conclusion is deduced from the quadratic optimal problem. However, they need solve a Riccati equation with undetermined R and Q. It is a quadratic equation. But the method given in Theorem 1 only solves an $(n-1)$-dimensional linear Lyapunov equation, this is simpler on the method.

Stabilization by CLF

If $V(x) = x^T Px$ is a CLF for the system (5), $\frac{\partial V}{\partial x} Ax = 2x^T PAx$, $\frac{\partial V}{\partial x} B = 2x^T PB$, the Sontag's universal formula gives the following stabilizing feedback (Sontag, 1983)

$$u = \begin{cases} -\frac{x^T PAx + \sqrt{(x^T PAx)^2 + 4(x^T PB)^4}}{x^T PB}, & x^T PB \neq 0, \\ 0, & x^T PB = 0. \end{cases} \tag{15}$$

However feedback (15) is not linear. In the linear system theory, we always desire to design a linear control. The following Theorem 2 gives a linear feedback which can link the poles of the closed-loop to the eigenvalues of C_β.

Let L_{-1} be the shift operator in R^n, i.e.,

$$L_{-1}x = L_{-1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ 0 \end{bmatrix}.$$

Clearly

$$L_{-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Without loss of the generality, the system considered in Theorem 2 takes the form of (6).

Theorem 2 If $V(x) = x^T Px$ is a CLF for the system (6), where $P = \begin{bmatrix} P_{n-1} & P_{12} \\ P_{12}^T & p_{22} \end{bmatrix}$ and $P_{12}^T = p_{22} \begin{bmatrix} \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}$. Then

$$u = -B^T (P + p_{22}^{-1} PL_{-1})x \tag{16}$$

can stabilize the system (6). Moreover, the poles of the closed-loop system are $-p_{22}$, and the $(n-1)$ of characteristic roots of C_β.

Proof: The closed-loop system combined by (6) and (16) is

$$\dot{x} = (A_c - BB^T (P + p_{22}^{-1} PL_{-1}))x. \tag{17}$$

Since

$$A_c - BB^T (P + p_{22}^{-1} PL_{-1}) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -p_{22}\beta_1 & -p_{22}\beta_2 - \beta_1 & \cdots & -p_{22} - \beta_{n-1} \end{bmatrix},$$

the characteristic polynomial of $A_c - BB^T (P + p_{22}^{-1} PL_{-1})$ is

$$H(\lambda) = (\lambda + p_{22})(\lambda^{n-1} + \beta_{n-1}\lambda^{n-2} + \cdots + \beta_2 \lambda + \beta_1).$$

The conclusion follows immediately.

Remark 5 Poles of the closed-loop system can be designed in C_β by Theorem 2, which in turn implies that we have the object to follow from the beginning.

Corollary 1 If $V(x) = x^T Px$ is a CLF for the system (5), then

$$u = -(B^T P + B^T p_{22}^{-1} PL_{-1})x - [G_{21} \quad \alpha_0]x \quad (18)$$

can stabilize the system (5). Moreover, the poles of the closed-loop system are $-p_{22}$, and the $(n-1)$ of characteristic roots of C_β.

Example 2

Continue to consider Example 1 again, it can obtain that poles of the open-loop system are $2.2403, -0.2876, 0.0237+1.2456i, 0.0237-1.2456i$. By (18), the feedback takes

$$u = [-25\ -53\ -34\ -12]x.$$

Then poles of the closed-loop system are

$$-4, -3, -2, -1.$$

The system becomes stable.

B. CLFs for A Multi-Input Controllable System

This subsection turns to consider the muti-input case.

Without loss of the generality, we assume that $rank(B) = m$ and (A, B) holds its Yokoyama canonical form (Yokoyama and Kinnen, 1973) provided that (A, B) is controllable.

$$A = \begin{bmatrix} 0 & [I_\nu \ 0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [I_2 \ 0] \\ -A_\nu & -A_{\nu-1} & \cdots & -A_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_1 \end{bmatrix} \quad (19)$$

where I_i, A_i are respectively $n_i \times n_i$ unit matrices and $m \times n_i$ real matrices, for $i = 2, 3, \cdots \nu$, and A_1 is an $n_1 \times n_1$ real matrix. $m = n_1 \geq n_2 \geq \cdots \geq n_\nu > 0$ are the controllability indices of (A, B). At the last equation B_1 is an $m \times m$ nonsingular matrix.

A, x are written in their block forms $A = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & -A_1 \end{bmatrix}$, $x^T = \begin{bmatrix} X_{n-m}^T & X_m^T \end{bmatrix}$, where

$$G_{11} = \begin{bmatrix} 0 & [I_\nu \ 0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [I_3 \ 0] \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$G_{12} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ [I_2 \quad 0] \end{bmatrix},$$

$G_{21} = \begin{bmatrix} -A_\nu & -A_{\nu-1} & \cdots & -A_2 \end{bmatrix}$, and
$X_{n-m}^T = \begin{bmatrix} x_1 & \cdots & x_{n-m} \end{bmatrix}$,
$X_m^T = \begin{bmatrix} x_{n-m+1} & \cdots & x_n \end{bmatrix}$.

At first, by an input transformation $u = B_1^{-1} u_1$, and a state feedback $u_1 = -\begin{bmatrix} G_{21} & -A_1 \end{bmatrix} x + u_2$, the system (19) is transformed into

$$\dot{x} = A_c x + B_c u_2 \quad (20)$$

where $A_c = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}$, and $B_c = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$.

We now study CLFs for the system (20).

Let P be a symmetric matrix. Divide P into a block form as follows

$$P = \begin{bmatrix} P_{n-m} & P_{12} \\ P_{12}^T & P_m \end{bmatrix}$$

where $P_{n-m} \in R^{(n-m)\times(n-m)}$, $P_m \in R^{m\times m}$ and $P_{12} \in R^{(n-m)\times m}$. Denote

$P_m^{-1} P_{12}^T = \begin{bmatrix} S_\nu & S_{\nu-1} & \cdots & S_2 \end{bmatrix}$, and $S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \end{bmatrix}$

where S_i, S_{i1}, and S_{i2} are respectively $n_1 \times n_i$, $n_2 \times n_i$, $(n_1 - n_2) \times n_i$ real matrices, for $i = 2, 3, \cdots, \nu$.

Denote

$$C_\beta = \begin{bmatrix} 0 & [I_\nu \ 0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [I_3 \ 0] \\ -S_{\nu 1} & -S_{\nu-11} & \cdots & -S_{21} \end{bmatrix}. \quad (21)$$

Consider the following Conditions:

H3 P_m is a positive definite matrix, and C_β is a Hurwitz matrix.

H4 $(P_{n-m} - P_{12} P_m^{-1} P_{12}^T) C_\beta + C_\beta^T (P_{n-m} - P_{12} P_m^{-1} P_{12}^T)$ is negative definite.

There are two Remarks to the above Conditions.

Remark 6 H4 implies $(P_{n-m} - P_{12} P_m^{-1} P_{12}^T)$ is a positive definite matrix provided that C_β is a Hurwitz matrix.

Remark 7 Since

$$\begin{bmatrix} I_{n-m} & -P_{12} P_m^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} P_{n-m} & P_{12} \\ P_{12}^T & P_m \end{bmatrix} \cdot \begin{bmatrix} I_{n-m} & 0 \\ -P_m^{-1} P_{12}^T & I_m \end{bmatrix} = \begin{bmatrix} P_{n-m} - P_{12} P_m^{-1} p_{12}^T & 0 \\ 0 & P_m \end{bmatrix},$$

H3 and H4 imply that P is a positive definite matrix.

By using the Yakoyama canonical form, the following Theorems can be established. We only state these results and omit their proofs.

Theorem 3 $V(x) = x^T Px$ is a CLF for the system (20) if and only if P satisfies Conditions H3 and H4.

The proof of Theorem 3 is exact to be the same as that of Theorem 1, and is omitted.

Denote

$$F = \begin{bmatrix} I_v & & & \\ & \ddots & & \\ & & I_2 & \\ & & & P_m^{-1} \end{bmatrix},$$

$$L_{-m} = \begin{bmatrix} 0 & [\,I_v \quad 0\,] & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & [\,I_2 \quad 0\,] \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Theorem 4 If $V(x) = x^T Px$ is a CLF for the system (20), where $P = \begin{bmatrix} P_{n-m} & P_{12} \\ P_{12}^T & P_m \end{bmatrix}$ and $P_{12}^T = P_m [\, S_v \quad \cdots \quad S_2 \,]$, C_β is denoted by (21). Then

$$u = -B_c^T(P + FPL_{-m})x \tag{22}$$

can stabilize the system (20). Moreover, the poles of the closed-loop system are m of eigenvalues of $-P_m$, and $(n-m)$ of characteristic roots of C_β.

The proof of Theorem 4 is similar to that of Theorem 2 and omitted too.

Remark 8 Since C_β is a real matrix, the $(n-m)$ poles assigned by (22) consist of a conjugate set. Moreover, P_m is a symmetric matrix the remaining m poles are all real.

C. The Inverse Problem of Optimization

In Remark 4, we mentioned that Freeman and Primbs (1996) proved that the positive definite solution of the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

can yields a CLF $V(x) = x^T Px$. It implies that the feedback

$$u = -B^T Px \tag{23}$$

is the solution of the optimization with the objective function

$$Min \quad J = \int_0^\infty (x^T Qx + u^T Ru)dt.$$

This subsection will verify if $V(x) = x^T Px$ is a CLF of the system (1), then P is the positive solution of a Riccati equation.

If $V(x) = x^T Px$ is a CLF of the system (1), we now consider the feedback

$$u = -cB^T Px \tag{24}$$

where c is a positive number to be determined. By the feedback (24), the closed-loop system becomes

$$\dot{x} = (A - cBB^T P)x. \tag{25}$$

The derivative of $V(x)$ along (25) is

$$\begin{aligned} \dot{V}(x) &= x^T(A^T P + PA - 2cPBB^T P)x \\ &= x^T(A^T P + PA)x - 2c \parallel B^T Px \parallel^2 \end{aligned} \tag{26}$$

where $\parallel B^T Px \parallel = \sqrt{x^T PBB^T Px}$ is the Euclidean norm of $B^T Px$. Since $V(x)$ is a CLF of the system (1), then

$$B^T Px = 0, x \neq 0 \Rightarrow x^T(A^T P + PA)x < 0. \tag{27}$$

On the other hand, when $B^T Px \neq 0$, the signal of $\dot{V}(x)$ will be the same as that of $-c \parallel B^T Px \parallel^2$ provided that c is large enough. Thus we can find a $c > 0$ such that for every $x \neq 0$,

$$x^T(A^T P + PA - 2cPBB^T P)x < 0,$$

i.e., the matrix

$$A^T P + PA - 2cPBB^T P$$

is negative definite.

It implies there exists a positive definite matrix Q such that P is the positive solution of the Riccati equation

$$A^T P + PA - PB(2c)B^T P + Q = 0.$$

Thus we obtain the following Theorem.

Theorem 5 If $V(x)$ is a CLF of the system (1), then there is a $c > 0$ such that P is the uniquely positive definite solution of the Riccati equation

$$A^T P + PA - PB(2c)B^T P + Q = 0$$

where Q is a positive definite matrix. Moreover,

$$u = -B^T Px$$

is the solution of the optimization of

$$Min \quad J = \int_0^\infty (x^T Qx + \frac{1}{2c}u^T u)dt.$$

Remark 9 By Theorem 5, we can conclude that $V(x) = x^T Px$ is a CLF of the system (1), then P is the positive solution of a Riccati equation. It means the the condition given by Freeman and Primbs (1996) is also necessary.

IV. UNCONTROLLABLE CASE

If the system (1) is non-completely controllable, i.e. the rank k of controllability matrix is less than n, then there exists a coordinate transformation $\bar{x} = T_1 x$ such that the system (1) is decomposed into

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u \tag{28}$$

and $(\bar{A}_c, \bar{B}_c)$ is controllable by Chen (1984).

Since $(\bar{A}_c, \bar{B}_c)$ is controllable, there exists a state feedback $u_c = K_c\bar{x}_c$, such that the eigenvalues of $\bar{A}_c + \bar{B}_cK_c$ are different from those of $\bar{A}_{\bar{c}}$. Then the matrix equation $(\bar{A}_c + \bar{B}_cK_c)F_1 - F_1\bar{A}_{\bar{c}} = \bar{A}_{12}$ has a unique solution F_1. By a state feedback $u = [K_c \quad 0]\bar{x} + v$ and a coordinate transformation $z = T_2\bar{x}$ where $T_2 = \begin{bmatrix} I_k & F_1 \\ 0 & I_{n-k} \end{bmatrix}$, the system (28) is transformed into a block digonal form as follows

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c + \bar{B}_cK_c & 0 \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_c \\ z_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} v \tag{29}$$

and $(\bar{A}_c + \bar{B}_cK_c, \bar{B}_c)$ is controllable.

Theorem 6 If the system (1) is non-completely controllable, then the system (1) holds a CLF if and only if the uncontrollable subsystem

$$\dot{z}_{\bar{c}} = \bar{A}_{\bar{c}}z_{\bar{c}} \tag{30}$$

is stable.

Proof: (Sufficiency) Since

$$\dot{z}_c = (\bar{A}_c + \bar{B}_cK_c)z_c + \bar{B}_cv \tag{31}$$

is completely controllable, the subsystem (31) holds a quadratic CLF $V(z_c) = z_c^TP_cz_c$ by Theorem 3. In view of the system (30) being stable, by Lyapunov Theorem, there exists a positive definite matrix $P_{\bar{c}}$, such that

$$\bar{A}_{\bar{c}}^TP_{\bar{c}} + P_{\bar{c}}\bar{A}_{\bar{c}} = -Q_{\bar{c}}, \tag{32}$$

where $Q_{\bar{c}} \in R^{(n-k)\times(n-k)}$ is an arbitrary positive definite matrix. It is direct to verify that $V(z) = z_c^TP_cz_c + z_{\bar{c}}^TP_{\bar{c}}z_{\bar{c}}$ is a CLF for the system (29). Then $V_x(x) = V(T_2T_1x)$ is a CLF for the system (1) by Propositions 1 and 3.

(Necessity) In the light of Theorem 2.5 in Sontag (1983), the system (1) is stabilizable if these exists a CLF. By linear system theory, the subsystem (30) has to be stable. ⋄

To end this section, we give the following Remark to show the relation between a CLF and a quadratic CLF of the linear system.

Remark 10 For any linear system (1), if the rank of its controllability matrix is n, then it is completely controllable. Thus there exists a quadratic CLF by Theorem 3. If the rank of its controllability matrix is less than n, then there exists a quadratic CLF $V_x(x) = V(T_2T_1x)$ for the system (1) by Theorem 6. In conclusion, for the linear system (1), if there exists a CLF then there exists a quadratic CLF.

V. THE ALGORITHM FOR THE CONSTRUCTION OF CLFS

The section concludes the algorithm for the construction of the quadratic CLFs of the system (1) from the above sections. We always require that the $rank(B) = m$ in (1).

Algorithm(Quadratic CLF construction)

Step1 Taking a controllability decomposition such that (1) is transformed into (29). From this step, we obtain two coordinate transformation matrices T_1 and T_2, as well as the dimension of controllable subsystem $(\bar{A}_c, \bar{B}_c)$.

Step2 Transforming the controllable subsystem into its Yokoyama canonical form. The controllability indices of $(\bar{A}_c, \bar{B}_c)$ are obtained from this step. Denote the indices to be $m = n_1 \geq n_2 \geq \cdots \geq n_\nu$, and $n_1 + n_2 + \cdots + n_\nu = k$. The transformation in this step is denoted by T_c.

Step3 Choosing $S_{i1} \in R^{n_2\times n_i}$, for $i = 2, 3, \cdots \nu$, such that the matrix C_β defined in (21) is Hurwitz.

Step4 Choosing $S_{i2} \in R^{(n_1-n_2)\times n_i}$ for $i = 2, 3, \cdots \nu$, arbitrarily and a positive definite matrix $P_m \in R^{m\times m}$. Calculating $P_{12}^T = P_m \begin{bmatrix} S_\nu & S_{\nu-1} & \cdots S_2 \end{bmatrix}$ where $S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \end{bmatrix}$ for $i = 2, 3, \cdots \nu$.

Step5 Choosing a positive definite matrix $Q \in R^{(k-m)\times(k-m)}$, and solving the Lyapunov equation $MC_\beta + C_\beta^TM = -Q$. By the Lyapunov Theorem, the solution M is positive definite because C_β is Hurwitz.

Step6 Calculating $P_{k-m} = M + P_{12}P_m^{-1}P_{12}^T$. After the step, we can construct the positive definite matrix $P_c = \begin{bmatrix} P_{k-m} & P_{12} \\ P_{12}^T & P_m \end{bmatrix}$.

Step7 For the uncontrollable subsystem $(\bar{A}_{\bar{c}} \quad \bar{B}_{\bar{c}})$, solving the Lyapunov equation $\bar{A}_{\bar{c}}^TP_{\bar{c}} + P_{\bar{c}}\bar{A}_{\bar{c}} = -Q_{\bar{c}}$, where $Q_{\bar{c}} \in R^{(n-k)\times(n-k)}$ is an arbitrary positive definite matrix. From the step we obtain the CLF for (29). The CLF is $V(z) = z_c^TT_c^TP_cT_cz_c + z_{\bar{c}}^TP_{\bar{c}}z_{\bar{c}}$.

Step8 Calculating $V_x(x) = V(T_2T_1x)$. $V_x(x)$ is a CLF of the system (1).

VI. CONCLUSION

This paper develops a systematic method by which CLF of linear systems can be constructed. It proves that the method can provide all quadratic CLFs for a given linear system. Moreover, by using the CLF a linear feedback is established to stabilize the linear system. It not only can stabilize the linear system but also assign poles of the closed-loop system in the position designed that satisfies Remark 8.

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