DECOUPLING WITH STABILITY OF LINEAR MULTIVARIABLE SYSTEMS: AN ALGEBRAIC APPROACH

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***Abstract*– The result that a linear multivariable system is decouplable with stability if and only if its associated stable interactor is diagonal, is proved in this paper using an algebraic approach. As it will be shown, this condition is actually equivalent to the coincidence between the infinite and unstable global structure (infinite and unstable zeros) and the row infinite and unstable structure of the system. Two procedures are presented to compute a state feedback which decouples the system with stability, the first one based on the solution of a polynomial matrix equation, and the second one based on the static left kernel of a strictly proper rational matrix. Illustrative examples are also presented.**

***Keywords*– Linear systems, Decoupling, Stability, Infinite structure, Algebraic approach.**

I. INTRODUCTION

Roughly speaking, decoupling of dynamic systems implies that each input of the system influences one and only one output without affecting the others. From the practical point of view, it is of interest to achieve decoupling because it is often desirable to control the outputs of the system independently.

The decoupling of linear multivariable systems by static state feedback has been extensively studied since the 1960's. In the case of linear systems with the same number of inputs and outputs (square systems), this problem was solved by Falb and Wolovich (1967). A structural solution in terms of the infinite zero structure of the system is presented in (Descusse and Dion, 1982).

The stability issue has necessarily to be considered in the problem formulation, since stability is a priority in the performance of any dynamical system. Decoupling with stability of square linear multivariable systems by static state feedback was solved by Martínez and Malabre (1994) using a geometric approach, and the solution is expressed in terms of the infinite and unstable contents of the system. The decoupling problem with stability of nonsquare systems is considered in (Ruiz-León *et al.*, 1995) using a polynomial equation approach, and a solution to this problem is reported. However, the necessary and sufficient conditions presented in this reference to solve the problem are implicit in the sense that the existence of a solution is stated in terms of the existence of a biproper and bistable rational matrix with certain properties, thus restricting a great deal the applicability of this result. Nevertheless, when these conditions are particularized to the case of square systems, a solution can be obtained in terms of the diagonality of the stable interactor of the system.

In this paper, using an algebraic approach, it is proved that the decoupling problem with stability has a solution if and only if the stable interactor of the system is a diagonal matrix. An important result, shown in our development, is the fact that the action of a state feedback on a stable system which preserves internal stability can be represented as a biproper and bistable matrix postmultiplying the system transfer function matrix. Thus, suitable factorizations of the system transfer function over the ring of proper and stable rational functions will provide the structural information to state the solution to the problem, namely, the global and row infinite and unstable structure. It is also shown how the stable interactor of the system displays this information.

This work complements the solution presented in (Martínez and Malabre, 1994) using a geometric approach. It is also proved here that both solutions are equivalent. Besides being an appealing alternative of solution, our approach has the advantage of providing a method to find a state feedback which solves the problem. Two procedures are presented to compute a state feedback which achieves decoupling with stability, the first one based on the solution of a polynomial matrix equation, and the second one based on the static left kernel of a strictly proper rational matrix.

This paper is organized as follows. The problem statement is presented in Section II. In Section III the infinite and unstable zero structure of a linear system is introduced via suitable factorizations of the system transfer function matrix, while the stable interactor is presented in Section IV, as well as a result about feedback realization of precompensators. The main results of this paper and two illustrative examples are presented in Section V. We end with some conclusions.

II. PROBLEM STATEMENT

We will first introduce some notation and basic definitions. Through this work, $\mathbb{R}$ will denote the field of real numbers, $\mathcal{C}$ will stand for the complex plane and $\mathcal{C}_-$ for the open left half complex plane. The set of polynomials in the variable "s" will be denoted by $\mathbb{R}[s]$ and $\mathbb{R}(s)$ represents the set of rational functions, i.e. rational fractions of the form $f(s) = a(s)/b(s)$, where $a(s)$ and $b(s)$ are coprime polynomials, and $b(s) \neq 0$. The rational function $f(s) = a(s)/b(s)$ is said to be proper if deg $b(s) \geq$ deg $a(s)$, strictly proper if deg $b(s) >$ deg $a(s)$, and proper and stable if it is proper, and its poles lie in the open left half complex plane $\mathcal{C}_-$. The set of proper rational functions will be denoted by $\mathbb{R}_p(s)$, and the set of proper and stable rational functions will be denoted by $\mathbb{R}_{ps}(s)$. The set of matrices of dimensions $p \times n$ with elements in $\mathbb{R}$, $\mathbb{R}_p(s)$ and $\mathbb{R}_{ps}(s)$, will be denoted, respectively, by $\mathbb{R}^{p\times n}$, $\mathbb{R}_p^{p\times n}(s)$ and $\mathbb{R}_{ps}^{p\times n}(s)$. A unimodular matrix is a nonsingular polynomial matrix whose inverse is also polynomial, a biproper matrix is a nonsingular proper rational matrix whose inverse is also proper, and a biproper and bistable matrix is a nonsingular proper and stable rational matrix whose inverse is also proper and stable.

We consider in this work linear multivariable and controllable systems with the same number of inputs and outputs, described in state space form by the equations

$$(A, B, C) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^p$ are, respectively, the state, input and output vectors of the system.

The system (A, B, C) is said to be row by row decouplable with stability by static state feedback if there exists a state feedback

$$(F, G): \quad u(t) = Fx(t) + Gv(t)$$

where $F \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{p\times p}$ are constant matrices with G nonsingular (regular static state feedback), and $v(t)$ is a new input vector, such that the input $v_i(t)$ controls the output $y_i(t)$, $i = 1, \ldots, p$, without affecting the other outputs, and the closed-loop system $(A+BF, BG, C)$ is internally stable, i.e. the eigenvalues of the matrix $(A + BF)$, which correspond to the modes of the closed-loop system, are located in the open left half complex plane $\mathcal{C}_-$.

From the input-output point of view, the previous formulation is equivalent to the existence of a state feedback (F, G) such that the transfer function $T_{F,G}(s)$ of the closed-loop system $(A + BF, BG, C)$ is of the form

$$\begin{aligned} T_{F,G}(s) &= C(sI - A - BF)^{-1}BG \\ &= \mathrm{diag}\{w_1(s), \ldots, w_p(s)\} =: W(s) \end{aligned} \quad (1)$$

where $w_i(s) \neq 0$, $i = 1, \ldots, p$, are strictly proper and stable rational functions, and the closed-loop system $(A + BF, BG, C)$ is internally stable.

We can suppose without loss of generality that the system (A, B, C) is internally stable, i.e. the eigenvalues of matrix A are located in $\mathcal{C}_-$. If this was not the case, there always exists a preliminary state feedback which will make the system stable, since we are considering that (A, B, C) is controllable. Thus, the transfer function matrix of the system

$$T(s) = C(sI - A)^{-1}B$$

can be considered as a strictly proper and stable rational matrix.

For simplicity we will consider that (A, B, C) is also observable. This consideration is in order that all finite zeros of the system appear in the system transfer function $T(s)$, and the conditions for decoupling can be directly tested from $T(s)$. Notice that this can always be supposed, since if the system is not observable, there always exists a preliminary state feedback which will make it observable, again due to the controllability condition of the system.

The system (A, B, C) will be supposed to be right invertible, which is a necessary condition for decoupling. This implies that the system transfer function $T(s)$ is nonsingular.

III. THE STRUCTURE OF THE SYSTEM

The solution to the decoupling problem with stability is related to the infinite and unstable zero structure of the system. This information can be obtained from a canonical form of the transfer function $T(s)$ over the ring of proper and stable rational functions, and it is actually a combination of the finite unstable zeros and the infinite zeros of the system. We will first recall the finite and infinite structure of the system, defined here from the Smith-McMillan and the Smith-McMillan form at infinity of the transfer function $T(s)$, and then present the infinite and unstable structure of the system. For the definition and properties of the Smith-McMillan form, as well as for basic concepts on algebraic theory of linear systems, see for instance (Kailath, 1980).

III.1 Finite structure

The finite structure of the system (finite poles and zeros) can be defined from the Smith-McMillan form of the system transfer function $T(s)$.

Let $T(s)$ be the transfer function of the system (A, B, C). Then, there exist unimodular matrices $U_1(s)$ and $U_2(s)$ such that

$$U_1(s)T(s)U_2(s) = M(s)$$

where

$$M(s) = \operatorname{diag}\left\{ \frac{\epsilon_i(s)}{\psi_i(s)} \right\}_{i=1}^{p} \tag{2}$$

is the Smith-McMillan form of $T(s)$, and $\{\epsilon_i(s),\ \psi_i(s)\}$, $i = 1, \ldots, p$, are coprime monic polynomials, uniquely determined by $T(s)$, satisfying the divisibility conditions

$$\epsilon_i(s) \mid \epsilon_{i+1}(s), \quad \text{and} \quad \psi_{i+1}(s) \mid \psi_i(s), \quad i = 1, \ldots, p-1. \tag{3}$$

The Smith-McMillan form $M(s)$ reveals the finite structure of the system: the roots of the polynomials $\epsilon_i(s)$ are the (transmission) finite zeros of the system, and the roots of $\psi_i(s)$ are the finite poles of the system.

III.2 Infinite structure

The infinite structure of the system can be defined from the Smith-McMillan form at infinity of the system transfer function $T(s)$ as follows (Vardulakis, 1991).

Let $T(s)$ be the transfer function of the system (A, B, C). Then, there exist biproper matrices $B_1(s)$ and $B_2(s)$, and a unique matrix $M_\infty(s)$, known as the Smith-McMillan form at infinity of $T(s)$, such that

$$B_1(s)T(s)B_2(s) = M_\infty(s) = \operatorname{diag}\left\{ \frac{1}{s^{n'_i}} \right\}_{i=1}^{p} \tag{4}$$

and $\{n'_1, \cdots, n'_p\}$ are positive integers satisfying

$$n'_i \leq n'_{i+1}, \qquad i = 1, \ldots, p-1. \tag{5}$$

The matrix $M_\infty(s)$ displays the infinite structure of the system, the integers $\{n'_1, \cdots, n'_p\}$ being the orders of the infinite zeros of the system.

The biproper matrices $B_1(s)$ and $B_2(s)$ in (4) represent, respectively, elementary row and column operations on $T(s)$ over the ring of proper rational functions $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$. Elementary row operations on $T(s)$ over $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$ are defined as follows:

i) changing the i-th and j-th rows of $T(s)$,
ii) multiplying the i-th row of $T(s)$ by a unit of $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$,
iii) adding to the i-th row of $T(s)$, the j-th row ($i \neq j$) multiplied by an element of $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$.

Elementary column operations are defined with the obvious changes.

The row infinite zero orders of the system (A, B, C), denoted as $\{n_1, \cdots, n_p\}$, are the infinite zero orders of the subsystems (A, B, C_i), where C_i is the i-th row of matrix C. Then, n_i, $i = 1, \ldots, p$, can be obtained from the Smith-McMillan form at infinity of the i-th row of the system transfer function $T(s)$.

Following (Martínez and Malabre, 1994), the content at infinity of the system is defined as

$$C_\infty(A, B, C) = \sum_{i=1}^{p} n'_i$$

and the row content at infinity corresponds to the row infinite zero orders, i.e.

$$C_\infty(A, B, C_i) = n_i, \qquad i = 1, \ldots, p.$$

III.3 Infinite and unstable structure

A key role in the decoupling problem with stability is played by the infinite and unstable zero structure of the system, introduced below. This information is displayed by the Smith form of the system transfer matrix over the set of proper and stable rational functions $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$. The set $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$ is known to be an Euclidean ring (Vidyasagar, 1985), the degree of a proper and stable rational function $f(s) \in \mathbb{R}_{\mathrm{ps}}(\mathrm{s})$, hereafter denoted $\deg_{\mathrm{ps}} f(s)$, taken as the number of infinite plus unstable zeros of $f(s)$.

Lemma 1 Let $T(s)$ be the transfer function of the system (A, B, C). Then, there exist biproper and bistable matrices $V_1(s)$ and $V_2(s)$, and a matrix $Z(s)$, unique up to units of $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$, such that

$$V_1(s)T(s)V_2(s) = Z(s) = \operatorname{diag}\{z_i(s)\}_{i=1}^{p} \tag{6}$$

and $z_i(s)$ are proper and stable rational functions satisfying the divisibility conditions in $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$,

$$z_i(s) \mid z_{i+1}(s), \qquad i = 1, \ldots, p-1. \tag{7}$$

Proof. It follows from the fact that $Z(s)$ is the Smith form of the proper and stable rational matrix $T(s)$ over the ring $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$. ∎

Let $M(s)$ be the Smith-McMillan form of $T(s)$ and factorize the polynomials $\epsilon_i(s)$ as

$$\epsilon_i(s) = \epsilon_i^-(s)\epsilon_i^+(s), \qquad i = 1, \ldots, p,$$

where $\epsilon_i^+(s)$ contains the roots of $\epsilon_i(s)$ outside the open left-half complex plane $\mathcal{C}_-$. Then, the proper and stable rational functions $z_i(s)$ in (6) have the form

$$z_i(s) = \frac{\epsilon_i^+(s)}{\pi^{\gamma_i}}, \qquad i = 1, \ldots, p, \tag{8}$$

where $\pi := s + \beta$ is a stable term (i.e. $-\beta \in \mathcal{C}_-$), and $\gamma_i := n'_i + \deg \epsilon_i^+(s)$, n'_i being the orders of the infinite zeros of the system.

The presence of some of the elements of $M(s)$ and $M_\infty(s)$ in $Z(s)$ is not at all unexpected. The matrix $Z(s)$ contains the information of $T(s)$ concerning the unstable and infinite zeros of the system, which can be obviously deduced combining the information about the finite zeros (given by $M(s)$) and the infinite zeros (given by $M_\infty(s)$) of the system.

The biproper and bistable matrices $V_1(s)$ and $V_2(s)$ in (6) represent, respectively, elementary row and column operations on $T(s)$ over the ring of proper and stable rational functions $\mathbb{R}_{\rm ps}(s)$.

Since we are considering that the system (A, B, C) is controllable and observable, then the so-called unstable content of the system (Martínez and Malabre, 1994), which is the information related to the unstable zeros of the system, is given by

$$C^+(A,B,C) = \sum_{i=1}^{p} \deg \epsilon_i^+(s).$$

The row unstable content of the system is the information related to the unstable zeros of the subsystems (A, B, C_i).

IV. DECOUPLING WITH STABILITY

Let (F, G) be a regular static state feedback applied on the stable system (A, B, C), such that the closed-loop system $(A + BF, BG, C)$ is internally stable. The closed-loop transfer function is given by

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG.$$

After some manipulations on the last equation, we obtain

$$\begin{aligned} T_{F,G}(s) &= C(sI-A)^{-1}B[I - F(sI-A)^{-1}B]^{-1}G \\ &= T(s)R(s) \end{aligned}$$

where $T(s) = C(sI - A)^{-1}B$ is the transfer function of the system (A, B, C), and

$$R(s) := [I - F(sI - A)^{-1}B]^{-1}G.$$

Since the closed-loop system is supposed to be stable, then $R(s)$ must be clearly a proper and stable rational matrix. Further, from

$$\begin{aligned} R^{-1}(s) &= G^{-1}[I - F(sI - A)^{-1}B] \\ &= \tfrac{1}{\det(sI-A)} G^{-1}[\det(sI - A)I - F \ \mathrm{Adj}(sI - A)B] \end{aligned}$$

it can be seen that $R^{-1}(s)$ is also proper and stable, since (A, B, C) is stable. Then, we have the following result.

Lemma 2 The effect of a regular static state feedback (F, G) which preserves internal stability can be represented in transfer function terms as a biproper and bistable matrix postmultiplying the system transfer function $T(s)$. ■

This fact, in its turn, establishes a natural restriction on the type of feedback we can use while trying to achieve decoupling with stability: For our purposes, the state feedback (F, G) will be said to be an admissible state feedback if its effect on the system (A, B, C) can be represented as a biproper and bistable precompensator $R(s)$ acting on the system transfer function $T(s)$.

This can be considered as the matrix interpretation of the fact that we are neither allowed to introduce unstable poles nor to cancel out unstable zeros in order to keep the internal stability of the closed-loop system.

At this stage, it is important to consider the information of the system which remains invariant under the action of biproper and bistable compensation, and consequently, invariant under the action of an admissible state feedback.

IV.1 The stable interactor

Since the action of an admissible state feedback on (A, B, C) can be represented as multiplication of $T(s)$ on the right by a biproper and bistable matrix, the information of the system that is invariant under such a feedback is contained in the column Hermite form of $T(s)$ over the ring $\mathbb{R}_{\rm ps}(s)$ (Dion and Commault, 1988; Ruiz-León *et al.*, 1995).

Lemma 3 Let $T(s)$ be the transfer function of (A, B, C). Then, there exist a biproper and bistable matrix $V(s)$ and a nonsingular lower triangular matrix $\Phi_s^{-1}(s) \in \mathbb{R}_{\rm ps}^{p\times p}(s)$, unique up to units of the ring $\mathbb{R}_{\rm ps}(s)$, such that

$$T(s)V(s) = \Phi_s^{-1}(s) = \begin{bmatrix} \varphi_{11}(s) & & (0) \\ \vdots & \ddots & \\ \varphi_{p1}(s) & \dots & \varphi_{pp}(s) \end{bmatrix} \quad (9)$$

where the rational functions $\varphi_{ij}(s) \in \mathbb{R}_{\rm ps}(s)$ satisfy, for $i > j$,

$$\varphi_{ij}(s) = 0, \ \text{ or } \ \deg_{\rm ps} \varphi_{ij}(s) < \deg_{\rm ps} \varphi_{ii}(s), \quad (10)$$

and they are of the form

$$\varphi_{ii}(s) = \frac{\alpha_{ii}(s)}{\pi^{k_{ii}}} \quad (11)$$

$$\varphi_{ij}(s) = \frac{\alpha_{ij}(s)}{\pi^{k_{ij}}}, \quad (12)$$

where $\alpha_{ii}(s)$ is a polynomial with only unstable roots (antistable polynomial), $\pi = s + \beta$ is a stable term, $\alpha_{ij}(s) \in \mathbb{R}[s]$ is a polynomial, and k_{ii}, k_{ij} are positive integers. The matrix $V(s)$ represents elementary column operations on $T(s)$ over $\mathbb{R}_{\rm ps}(s)$. The positive real number β results from the column operations in (9) and it is inherited from $T(s)$, otherwise it is arbitrary. ■

The rational matrix $\Phi_s(s)$, which is the inverse of $\Phi_s^{-1}(s)$, is known as the stable interactor of the system. This matrix is also called generalized interactor because, in some sense, it could be considered as a generalization of the classical system interactor $\Phi(s)$ (Wolovich and Falb, 1976). Notice, as stated in Lemma 3, that the matrix $\Phi_s(s)$ is not unique but it is unique up to units of the ring $\mathbb{R}_{\rm ps}(s)$. For a fixed $\pi = s+\beta$, the matrix $\Phi_s(s)$ is unique, and considering this is why we call it "the" stable interactor of the system. Actually, the algebraic properties of $\Phi_s(s)$ presented afterwards do not depend on the choice of β, nor the results stated in the next sections based on these properties.

The matrix $\Phi_s(s)$ is in general a rational matrix having only unstable poles. This can be seen from the fact that the numerator of the determinant of $\Phi_s^{-1}(s)$ is the product of the antistable polynomials $\alpha_{ii}(s)$, $i = 1, \ldots, p$. Observe that if (A, B, C) has no unstable zeros, then $\Phi_s(s)$ is a polynomial matrix.

While the classical system interactor $\Phi(s)$ is a polynomial matrix containing the infinite structure of the system that can not be modified by a state feedback (Wolovich and Falb, 1976), the stable interactor is a state feedback invariant containing the infinite and unstable structural information of the system.

Remark 1 Let n_i be the infinite zero order of the i-th row of $T(s)$ and let d_i denotes the number of unstable zeros (with multiplicities included) of the same row of $T(s)$. Because of property (10), and since $V(s)$ in (9) is biproper and bistable, it can be seen that if the diagonal entry $\varphi_{ii}(s)$ is the only element different from zero of the i-th row of $\Phi_s^{-1}(s)$, then

$$n_i + d_i = \deg_{\rm ps}\, \varphi_{ii}(s),$$

otherwise

$$n_i + d_i < \deg_{\rm ps}\, \varphi_{ii}(s).$$

The last equation follows from the fact that if $\varphi_{ii}(s)$ is not the only element different from zero of the i-th row of $\Phi_s^{-1}(s)$, then there exists an element $\varphi_{ij}(s)$ in the same row such that

$$n_i + d_i = \deg_{\rm ps}\, \varphi_{ij}(s) < \deg_{\rm ps}\, \varphi_{ii}(s).$$

IV.2 Feedback realization of precompensators

A given proper compensator $P(s)$ is said to be feedback realizable on the system (A, B, C) if there exists a state feedback (F, G) such that

$$P(s) = [I - F(sI - A)^{-1}B]^{-1}G.$$

The following result, used in the proof of Theorem 1, states the conditions for a proper compensator to be realizable.

Lemma 4 (Hautus and Heymann, 1978) Let the matrices $N_1(s)$ and $D(s)$ be a right coprime matrix fraction description (MFD) of the system (A, B, I_n), and let $P(s)$ be a nonsingular compensator. Then $P(s)$ is state feedback realizable on (A, B, C) if and only if

- $P(s)$ is biproper, and
- $P^{-1}(s)D(s)$ is a polynomial matrix. ■

V. MAIN RESULTS

V.1 Solution to decoupling with stability

The next result presents the necessary and sufficient conditions for the decoupling problem with stability to have a solution.

Theorem 1 The controllable and stable square system (A, B, C) is decouplable with stability if and only if its associated stable interactor $\Phi_s(s)$ is a diagonal matrix.

Proof. Necessity. Suppose that (A, B, C) is decouplable with stability. Then there exists a state feedback (F, G) such that

$$\begin{aligned} T_{F,G}(s) &= C(sI - A - BF)^{-1}BG \\ &= T(s)[I - F(sI - A)^{-1}B]^{-1}G = W(s). \end{aligned}$$

Since $[I - F(sI - A)^{-1}B]^{-1}G$ is a biproper and bistable matrix, and $W(s) = \text{diag}\{w_1(s), \ldots, w_p(s)\}$ is proper and stable, then it follows that $\Phi_s(s)$ is diagonal.

Sufficiency. To prove this part, we will show that the biproper and bistable matrix $V(s)$ in (9) is feedback realizable. Let $N_1(s)$ and $D(s)$ be a right coprime MFD of (A, B, I_n) with $D(s)$ column reduced. According to Lemma 4, $V(s)$ will be proved to be feedback realizable if the product $V^{-1}(s)D(s)$ is polynomial.

From (9) we have

$$T(s)V(s) = CN_1(s)D^{-1}(s)V(s) = \Phi_s^{-1}(s),$$

and from this, we get

$$\Phi_s(s)CN_1(s) = V^{-1}(s)D(s). \tag{13}$$

Since the left hand side of the last equation has only unstable poles, and the right hand side has only stable poles, then it follows that $V^{-1}(s)D(s)$ is polynomial.

Given that $V^{-1}(s)D(s)$ is polynomial, and taking $D(s)$ column reduced, then there exists a constant solution X, Y with X nonsingular, to the polynomial matrix equation (Kučera and Zagalak, 1991)

$$XD(s) + YN_1(s) = V^{-1}(s)D(s). \tag{14}$$

Thus, a state feedback (F, G) which decouples the system (A, B, C) and preserves internal stability is obtained as $F = -X^{-1}Y$, $G = X^{-1}$. ■

Remark 2 From the fact that the biproper and bistable matrix $V(s)$ in (9) is feedback realizable, then $\Phi_s^{-1}(s)$ (diagonal or not) can be considered as the transfer function of the closed-loop system $(A + BF, BG, C)$, whose stable interactor is $\Phi_s(s)$. Thus, the matrix $V(s)$ can be regarded as the input-output representation of a static state feedback whose effect is a pole assignment, some poles of the system being placed at the positions of stable zeros producing cancellation, and the remaining ones being located at the stable position $s = -\beta$. In this way, $\Phi_s^{-1}(s)$ is the resulting transfer function of $(A + BF, BG, C)$. Compare with $\Phi^{-1}(s)$, the inverse of the classic interactor, where it can be considered that cancellation of all zeros is produced and the remaining poles are located at $s = 0$. From this point of view, the difference between $\Phi^{-1}(s)$ and $\Phi_s^{-1}(s)$ is that in the process of obtaining $\Phi_s^{-1}(s)$ we avoid the cancellation of unstable zeros in order not to produce internal instability.

The following result shows that the conditions presented in (Martínez and Malabre, 1994) are equivalent to the conditions of Theorem 1.

Theorem 2 The stable interactor of the system (A, B, C) is diagonal if and only if the following two conditions hold

$$C_\infty(A, B, C) = \sum_{i=1}^{p} C_\infty(A, B, C_i), \tag{15}$$

and

$$C^+(A, B, C) = \sum_{i=1}^{p} C^+(A, B, C_i). \tag{16}$$

where $C_\infty(A, B, C)$ ($C_\infty(A, B, C_i)$) is the content at infinity (row content at infinity) and $C^+(A, B, C)$ ($C^+(A, B, C_i)$) is the unstable content (row unstable content) of the system.

Proof. Necessity. Let $Z(s)$ be the Smith form of $T(s)$ over $\mathbb{R}_{ps}(s)$, and let $\Phi_s^{-1}(s)$ be the inverse of the stable interactor. From (6) and (9) it follows that

$$\deg_{ps}[\det\ Z(s)] = \deg_{ps}[\det\ \Phi_s^{-1}(s)].$$

Then, supposing that $\Phi_s(s)$ is diagonal we have that

$$\sum_{i=1}^{p}(n_i' + r_i) = \sum_{i=1}^{p}(n_i + d_i) \tag{17}$$

where n_i' (n_i) are the infinite (row infinite) zero orders of the system, d_i is the number of row unstable zeros, and $r_i := \deg\ \epsilon_i^+(s)$ where $\epsilon_i^+(s)$ are the antistable polynomials in (8).

Since the system is decouplable with stability ($\Phi_s(s)$ is diagonal), then it is also decouplable without stability, i.e. (Descusse and Dion, 1982)

$$\sum_{i=1}^{p} n_i' = \sum_{i=1}^{p} n_i$$

and from this and (17), we have

$$\sum_{i=1}^{p} r_i = \sum_{i=1}^{p} d_i$$

which are respectively equivalent to (15) and (16).

Sufficiency. From (15) and (16) it follows that

$$\sum_{i=1}^{p}(n_i' + r_i) = \sum_{i=1}^{p}(n_i + d_i).$$

From Remark 1, we have that

$$n_i + d_i \leq \deg_{ps}\ \varphi_{ii}(s),$$

equality holding in the case that $\varphi_{ii}(s)$ is the only element different from zero in the i-th row of $\Phi_s^{-1}(s)$.

Let us suppose that $\Phi_s^{-1}(s)$ is not diagonal, this means that

$$\sum_{i=1}^{p}(n_i + d_i) < \sum_{i=1}^{p} \deg_{ps}\ \varphi_{ii}(s)$$

i.e.

$$\sum_{i=1}^{p}(n_i + d_i) < \sum_{i=1}^{p}(n_i' + r_i)$$

contradicting our assumption. ■

V.2 Computation of the decoupling state feedback

Considering that the system (A, B, C) is decouplable with stability, two methods are presented in this section to compute a state feedback which decouples the system with stability.

Method 1. If the stable interactor of the system is a diagonal matrix, then the sufficiency part of Theorem 1 provides a procedure to find a state feedback which produces $\Phi_s^{-1}(s)$ as the transfer function of the closed-loop system for any choice of β. Actually, the poles of this diagonal closed-loop transfer function can be arbitrarily located in $\mathcal{C}_-$. To obtain the corresponding state feedback, find a constant solution X, Y with X nonsingular to the polynomial matrix equation

$$XD(s) + YN_1(s) = W^{-1}(s)CN_1(s) \tag{18}$$

where $W(s)$ is the diagonal closed-loop transfer function with desired poles, obtained from $\Phi_s^{-1}(s)$, and C, $N_1(s)$ and $D(s)$ are as before. Thus, a state feedback (F, G) which decouples the system (A, B, C) with closed-loop transfer function $W(s)$ and preserves internal stability is obtained as $F = -X^{-1}Y$, $G = X^{-1}$.

Method 2. Let $W(s)$ be the diagonal closed-loop transfer function with desired poles, obtained from $\Phi_s^{-1}(s)$, and define

$$Q(s) = T^{-1}(s)W(s) = Q_0 + \bar{Q}(s), \qquad (19)$$

where Q_0 is a constant matrix, and $\bar{Q}(s)$ is a strictly proper rational matrix.

Then, we search for matrices F and G such that

$$Q(s) = [I - F(sI - A)^{-1}B]^{-1}G. \qquad (20)$$

From the last equation, it can be seen that matrix G is given by

$$G = \lim_{s\to\infty} Q(s) = Q_0. \qquad (21)$$

To compute matrix F, let $[\,L \;\; E\,]$ be a basis for the left constant kernel of the matrix

$$\left[\begin{array}{c} (sI - A)^{-1}B \\ I - GQ^{-1}(s) \end{array}\right], \qquad (22)$$

where $L \in \mathbb{R}^{p\times n}$, $E \in \mathbb{R}^{p\times p}$, and E is nonsingular (such matrices exist if the system is decouplable with stability).

Then, matrix F is given by

$$F = -E^{-1}L. \qquad (23)$$

Remark 3 While Method 1 relies on the constant solution to a polynomial matrix equation, Method 2 uses a constant kernel of a strictly proper rational matrix. Method 2 has the advantage, in comparison to Method 1, that it is not necessary to obtain a matrix fraction description of the system with $D(s)$ column reduced.

V.3 Examples

The next two examples will illustrate the main results of this paper.

Example 1 Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -1 & 1 & 1 & 4 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix},$$

$$C = \left[\begin{array}{ccccc} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{array}\right],$$

whose transfer function matrix is

$$T(s) = C(sI - A)^{-1}B = \left[\begin{array}{cc} \frac{1}{(s+1)^2} & 0 \\ \frac{1}{(s+1)^4} & \frac{s-1}{(s+1)^3} \end{array}\right].$$

The stable interactor of this system is not a diagonal matrix, because the $(2, 1)$ entry of $T(s)$ can not be cancelled out by an elementary column operation over $\mathbb{R}_{ps}(s)$, thus the system is not decouplable with stability. The more we can do using elementary column operations over $\mathbb{R}_{ps}(s)$ is to obtain the Hermite column form $\Phi_s^{-1}(s)$ of $T(s)$, i.e. to reduce the degree of the $(2, 1)$ entry of $T(s)$ so as to be less than the degree of the $(2, 2)$ entry, which is equal to 3. Then, we have

$$T(s)V(s) = \left[\begin{array}{cc} \frac{1}{(s+1)^2} & 0 \\ \frac{1}{(s+1)^4} & \frac{s-1}{(s+1)^3} \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ \frac{s+3}{4(s+1)} & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc} \frac{1}{(s+1)^2} & 0 \\ \frac{1}{4(s+1)^2} & \frac{s-1}{(s+1)^3} \end{array}\right] = \Phi_s^{-1}(s)$$

where the last matrix is $\Phi_s^{-1}(s)$ for $\beta = 1$.

In terms of the infinite and unstable zeros of the system, and following the notation from Theorem 2, we have that

$$n_1 = n_2 = 2, \qquad n_1' = n_2' = 2$$
$$d_1 = d_2 = 0, \qquad r_1 = 0, \; r_2 = 1.$$

Then, it can be seen that the row infinite zeros coincide with the global infinite zeros, but not the unstable zeros, since $s = 1$ is an unstable global zero but it is not a row zero of $T(s)$. Thus, this system is decouplable, but not decouplable with stability. Indeed, the state feedback

$$F = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right], \quad G = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

produces the decoupled system $(A + BF, BG, C)$ whose transfer function is

$$T_{F,G}(s) = \left[\begin{array}{cc} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{array}\right]$$

but the closed-loop system is not internally stable.

Example 2 Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & -3 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right],$$

whose transfer function matrix is

$$T(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ \frac{s-1}{(s+1)^4} & \frac{s-1}{(s+1)^3} \end{bmatrix}.$$

The stable interactor for this system is a diagonal matrix

$$\Phi_s(s) = \begin{bmatrix} \frac{1}{\pi^2} & 0 \\ 0 & \frac{s-1}{\pi^3} \end{bmatrix}^{-1} = \begin{bmatrix} \pi^2 & 0 \\ 0 & \frac{\pi^3}{s-1} \end{bmatrix}$$

or equivalently, the row and global infinite and unstable zeros of the system coincide, and thus the system is decouplable with stability.

Following any of the two methods reported in the last section, we obtain the state feedback

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which produces the decoupled and stable closed-loop system $(A + BF, BG, C)$ whose transfer function is

$$T_{F,G}(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{s-1}{(s+1)^3} \end{bmatrix}.$$

Actually, this system can be decoupled with stability with closed-loop transfer function

$$T_{F,G}(s) = \begin{bmatrix} \frac{1}{(s+\lambda_1)(s+\lambda_2)} & 0 \\ 0 & \frac{s-1}{(s+\lambda_3)(s+\lambda_4)(s+\lambda_5)} \end{bmatrix}$$

where $\lambda_{1,2,3,4,5}$ are arbitrary positive real numbers.

VI. CONCLUSIONS

In this paper, we have presented a solution to the decoupling problem with stability of linear square multivariable systems by static state feedback using an algebraic approach. It has been shown that the problem has a solution if and only if the stable interactor is a diagonal matrix. This is a structural condition which is equivalent to the equality between the global infinite and unstable structure and the row infinite and unstable structure of the system. Two methods were presented to compute a state feedback which decouples the system with stability.

The assumptions made in this paper that the controllable system (A, B, C) is internally stable and observable are just in order to define properly the stable interactor of the system and to check the conditions for decoupling from the system transfer function $T(s)$, and there is no loss of generality in these assumptions. The decouplability of non-stable or non-observable systems can be also easily analyzed. Indeed, the structural conditions for decoupling with stability are independent of these properties of the system.

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