EXACT TRAVELLING WAVE SOLUTIONS TO THE GENERALIZED KURAMOTO-SIVASHINSKY EQUATION

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***Abstract*— By using a special transformation, the new exact travelling wave solutions to the generalized Kuramoto- Sivashinsky equation are obtained.**

***Keywords*— travelling wave solutions, solitary wave solutions, Kuramoto-Sivashinsky equation.**

I. INTRODUCTION

In this paper, we consider the generalized Kuramoto-Sivashinsky equation (Yang, 1994):

$$u_t + \beta u^\alpha u_x + \gamma u^\tau u_{xx} + \delta u_{xxxx} = 0\,, \qquad (1)$$

where $\alpha, \beta, \gamma, \delta, \tau \in R$ and $\alpha\beta\gamma\delta \neq \mathbf{0}$.

When $\alpha = \beta = 1$ and $\tau = 0$, (1) reduces to the original Kuramoto-Sivashinsky (K-S) equation. The K-S equation was derived by Kuramoto (1978) for the study of phase turbulence in the Belousov-Zhabotinsky reaction. An extension of this equation to two or more spatial dimensions was then given by Sivashinsky (1977, 1980) in the study of the propagation of a frame front for the case of mild combustion. The K-S equation represents one class of pattern formation equation (Yang, 1994; Temam, 1988), and it also serves as a good model of bifurcation and chaos (Abdel-Gawad & Abdusalam, 2001; Li and Chen 2001, 2002).

As far as the travelling wave solutions are concerned, one can always use the transform

$$u(x,t) = u(\xi)\,, \quad \xi = x - ct\,, \qquad (2)$$

where c is the wave velocity. The travelling wave solutions of (1) satisfy the following ordinary differential equation:

$$-cu' + \beta u^\alpha u' + \gamma u^\tau u'' + \delta u'''' = 0\,. \qquad (3)$$

In (Yang, 1994), using the ansatz (Bernoulli equation)

$$u' = au + bu^n\,, \qquad (4)$$

where $a, b, n \in R, ab < 0$ and $n \neq 1$, the exact travelling wave solution to (1) for $\alpha = 3\tau = 9$ was obtained. In this presentation, we further introduce the following ansatz:

$$u(\xi) = v^h(\xi)\,, \quad v' = av + bv^n\,, \qquad (5)$$

where $\mathbf{abh \neq 0, n \neq 1}$ and $\mathbf{ab < 0}$, and obtain a new exact solution for the equation.

From (5), one first gets

$$u(\xi) = \left[-\frac{a}{2b}\tanh\left(\frac{n-1}{2}a(\xi - c_0)\right) - \frac{a}{2b}\right]^{\frac{h}{n-1}}\,, \qquad (6)$$

in which c_0 is an arbitrary constant. If $h/(n-1) > 0$, (6) is the solitary wave solution connecting the two stationary states $u = 0$ and $u = (-\frac{a}{b})^{h/(n-1)}$ (Lu, *et al.*, 1993). So, the relative orbit is a heteroclinic orbit.

Repeating some differential calculations, one can obtain the following formulas:

$$v'' = (a + nbv^{n-1})v'\,, \qquad (7)$$

$$v'''' = [a^3 + a^2bn(n^2+n+1)v^{n-1} + 3ab^2n^2(2n-1)v^{2n-2} + b^3n(2n-1)(3n-2)v^{3n-3}]v'\,. \qquad (8)$$

$$u' = hv^{h-1}v'\,, \qquad (9)$$

$$u'' = [h^2av^{h-1} + hb(n+h-1)v^{n+h-2}]v'\,, \qquad (10)$$

$$u'''' = \{h^4a^3v^{h-1} + ha^2b(n+h-1)[h^2 + (n+h-1)\cdot(n+2h-1)]v^{n+h-2} + 3hab^2(n+h-1)^2(2n+h-2)v^{2n+h-3} + hb^3(n+h-1)(2n+h-2)(3n+h-3)v^{3n+h-4}\}\, v'\,. \qquad (11)$$

Then, by substituting the first formula of (5) and (9)-(11) into (3), one has

$$\{(-ch + \delta h^4a^3)v^{h-1} + \beta hv^{\alpha h+h-1} + \gamma h^2av^{\tau h+h-1} + \gamma hb(n+h-1)v^{n+(\tau+1)h-2} + \delta ha^2b(n+h-1)\cdot[3h^2 + 3h(n-1) + (n-1)^2]v^{n+h-2} + 3\delta hab^2(n+h-1)^2\cdot(2n+h-2)v^{2n+h-3} + \delta hb^3(n+h-1)(2n+h-2)\cdot$$

$$(3n+h-3)v^{3n+h-4}\Big\}\,v' = 0\,. \qquad (12)$$

By furthermore comparing the same orders of v, one can determine values of the parameters a, b, n and h. However, to consider all possible cases is rather complicated. In order to keep the presentation short, only the following interesting cases with $h = 1, n = 0$ and $n = 2$ are considered here.

II. CASE h=1

If $h = 1$, then $u(\xi) = v(\xi)$ and $u' = au + bu^n$, so that (12) is reduced to

$$(\delta a^3 - c) + \beta u^\alpha + \gamma a u^\tau + \gamma b n u^{\tau+n-1} +$$
$$\delta a^2 b n(n^2+n+1)u^{n-1} + 3\delta a b^2 n^2(2n-1)u^{2n-2} +$$
$$\delta b^3 n(2n-1)(3n-2)u^{3n-3} = 0\,. \qquad (13)$$

By comparing the same orders of u, one finds the following situations.

(1) $n = \frac{1}{2}$

(1a) $\tau = 0$, $\alpha = -\frac{1}{2}$

$$\delta a^3 - c + \gamma a = 0,\ \beta + \frac{1}{2}\gamma b + \frac{7}{8}\delta a^2 b = 0\,,$$

that is,

$$b = -\frac{8\beta}{4\gamma + 7\delta a^2}\,, \quad c = \delta a^3 + \gamma a\,. \qquad (14)$$

In (14), a is a parameter. One should choose a such that $ab < 0$. The same should be done for the similar cases below.

(1b) $\tau = -\frac{1}{2}$, $\alpha = -1$

$$\delta a^3 - c = 0\,, \quad \beta + \frac{1}{2}\gamma b = 0\,, \quad \gamma a + \frac{7}{8}\delta a^2 b = 0\,,$$

that is,

$$a = \frac{4\gamma^2}{7\delta\beta}\,, \quad b = -\frac{2\beta}{\gamma}\,,\ c = \delta a^3\,. \qquad (15)$$

(2) $n = \frac{2}{3}$

If $n = \frac{2}{3}$, then (13) can be translated into

$$(\delta a^3 - c) + \beta u^\alpha + \gamma a u^\tau + \frac{2}{3}\gamma b u^{\tau-\frac{1}{3}}$$
$$+\frac{38}{27}\delta a^2 b u^{-\frac{1}{3}} + \frac{4}{9}\delta a b^2 u^{-\frac{2}{3}} = 0\,.$$

The following results are immediate.

(2a) $\tau = 0$, $\alpha = -\frac{2}{3}$

$$\delta a^3 - c + \gamma a = 0\,, \quad \frac{2}{3}\gamma b + \frac{38}{27}\delta a^2 b = 0\,, \quad \beta + \frac{4}{9}\delta a b^2 = 0\,.$$

Therefore, one can easily find that

$$a = \pm 3\left(-\frac{\gamma}{19\delta}\right)^{\frac{1}{2}} (\gamma\delta < 0)\,, \quad b = \mp\frac{3}{2}\left(-\frac{\beta}{\delta a}\right)^{\frac{1}{2}}$$
$$(\delta\beta\, a < 0)\,, \quad c = \delta a^3 + \gamma a\,. \qquad (16)$$

(2b) $\tau = -\frac{1}{3}$, $\alpha = -\frac{1}{3}$

$$\delta a^3 - c = 0,\ \beta + \gamma a + \frac{38}{27}\delta a^2 b = 0\,, \quad \frac{2}{3}\gamma b + \frac{4}{9}\delta a b^2 = 0\,,$$

so,

$$a = \frac{9\beta}{10\gamma}\,, \quad b = -\frac{5\gamma^2}{3\delta\beta}\,, \quad c = \delta a^3\,. \qquad (17)$$

(2c) $\tau = -\frac{1}{3}$, $\alpha = -\frac{2}{3}$

$$\delta a^3 - c = 0\,, \quad \gamma a + \frac{38}{27}\delta a^2 b = 0\,, \quad \beta + \frac{2}{3}\gamma b + \frac{4}{9}\delta a b^2 = 0\,,$$

so,

$$a = \frac{90\gamma^2}{361\beta\delta}, \quad b = -\frac{57\beta}{20\gamma}, \quad c = \delta a^3\,. \qquad (18)$$

Besides $n = \frac{1}{2}$ and $n = \frac{2}{3}$, one has the following case.

(3) $\tau = n - 1$, $\alpha = 3n - 3$

For this case,

$$\delta a^3 - c = 0\,, \quad \gamma a + \delta a^2 b n(n^2+n+1) = 0\,,$$
$$\gamma b n + 3\delta a b^2 n^2(2n-1) = 0\,, \quad \beta + \delta b^3 n(2n-1)(3n-2) = 0$$

From the second and the third equations, it follows that $n = 4$. So, if and only if $n = 4, \alpha = 3\tau = 9$, there exist real number solutions for a, b and c, as

$$a = \frac{\gamma}{6}\left(\frac{5}{49\beta\delta^2}\right)^{\frac{1}{3}}, \quad b = \left(-\frac{\beta}{280\delta}\right)^{\frac{1}{3}}, \quad c = \frac{5\gamma^3}{10584\beta\delta}\,. \qquad (19)$$

This result is the same as that obtained in Yang (1994).

III. CASE n=0

For $n = 0$, (12) is reduced to

$$(\delta a^3 h^3 - c)v^{h-1} + \beta v^{\alpha h+h-1} + \gamma a h v^{\tau h+h-1} + \gamma b(h-1)\cdot$$
$$v^{\tau h+h-2} + \delta a^2 b(h-1)(3h^2-3h+1)v^{h-2} + 3\delta a b^2(h-1)^2\cdot$$
$$(h-2)v^{h-3} + \delta b^3(h-1)(h-2)(h-3)v^{h-4} = 0\,. \qquad (20)$$

After considering the coefficients of some orders of v, one has the following cases, with $h = 1$, $h = 2$ and $h = 3$, respectively.

(1) $h = 1$

There exists one and only one sub-case with $\alpha = \tau \neq 0$ for $h = 1$ (Note: $\alpha \neq 0$), as follows.

(1a) $\alpha = \tau \neq 0$,

$$a = -\frac{\beta}{\gamma}\,, \quad c = \delta a^3\,. \tag{21}$$

(2) $h = 2$

For $h = 2$, one also has a sub-case.

(2a) $\tau = 0$, $\alpha = -\frac{1}{2}$

For this sub-case, one has

$$7\delta a^2 b + \gamma b + \beta = 0, \quad 8\delta a^3 - c + 2\gamma a = 0\,.$$

Thus,

$$b = -\frac{\beta}{7\delta a^2 + \gamma}, \quad c = 8\delta a^3 + 2\gamma a\,. \tag{22}$$

(3) $h = 3$

For $h = 3$, (20) can be changed to

$$(27\delta a^3 - c)v^2 + \beta v^{3\alpha+2} + 3\gamma a v^{3\tau+2} + 2\gamma b v^{3\tau+1} + 38\delta a^2 b v + 12\delta a b^2 = 0\,, \tag{23}$$

and only three cases exist, as follows.

(3a) $\tau = -\frac{1}{3}$, $\alpha = -\frac{2}{3}$

Here,

$$27\delta a^3 - c = 0,\ 3\gamma a + 38\delta a^2 b = 0,\ \beta + 2\gamma b + 12\delta a b^2 = 0\,.$$

Therefore,

$$a = \frac{30\gamma^2}{361\delta\beta}\,, \quad b = -\frac{19\beta}{20\gamma}\,, \quad c = 27\delta a^3\,. \tag{24}$$

(3b) $\tau = -\frac{1}{3}$, $\alpha = -\frac{1}{3}$

It is clear that

$$27\delta a^3 - c = 0,\ \beta + 3\gamma a + 38\delta a^2 b = 0, 2\gamma b + 12\delta a b^2 = 0\,.$$

Hence,

$$a = \frac{3\beta}{10\gamma}\,, \quad b = -\frac{5\gamma^2}{9\delta\beta}\,, \quad c = 27\delta a^3\,. \tag{25}$$

(3c) $\tau = 0$, $\alpha = -\frac{2}{3}$

By the same reasoning, one has

$$27\delta a^3 - c + 3\gamma a = 0,\ 2\gamma b + 38\delta a^2 b = 0,\ \beta + 12\delta a b^2 = 0\,,$$

so a, b, c are as below:

$$a = \pm\left(-\frac{\gamma}{19\delta}\right)^{\frac{1}{2}}, \quad \left(\frac{\gamma}{\delta} < 0\right), b = \mp\left(-\frac{\beta}{12\delta a}\right)^{\frac{1}{2}}, \quad \left(\frac{\beta}{\delta a} < 0\right),\ c = 27\delta a^3 + 3\gamma a\,. \tag{26}$$

For $h \neq 1$, $h \neq 2$, and $h \neq 3$, a comparison between the corresponding terms in (20) gives only one case, as follows.

(4) $\alpha = 3\tau = -\frac{3}{h}$.

It follows that

$$\delta a^3 h^3 - c = 0\,, \quad \delta a^2 b(h-1)(3h^2 - 3h + 1) + \gamma a h = 0\,,$$
$$3\delta a b^2 (h-1)^2 (h-2) + \gamma b(h-1) = 0\,,$$
$$\delta b^3 (h-1)(h-2)(h-3) + \beta = 0\,.$$

The second and the third equations of the above system give $h = -\frac{1}{3}$. So, if and only if $h = -\frac{1}{3}, \alpha = 3\tau = 9$, the above system has real number solutions for a, b and c, as

$$a = -\frac{\gamma}{14\delta}\left(\frac{35\delta}{\beta}\right)^{\frac{1}{3}}, \ b = \frac{3}{2}\left(\frac{\beta}{35\delta}\right)^{\frac{1}{3}}, \ c = -\frac{\delta a^3}{27}\,. \tag{27}$$

IV. CASE n=2

Substituting $n = 2$ into (12) gives

$$(-c + \delta h^3 a^3)v^{h-1} + \delta a^2 b(h+1)(3h^2 + 3h + 1)v^h + 3\delta a b^2 (h+1)^2 (h+2) v^{h+1} + \delta b^3 (h+1)(h+2)(h+3) v^{h+2} + \gamma h a v^{\tau h + h - 1} + \gamma b(h+1) v^{\tau h + h} + \beta v^{\alpha h + h - 1} = 0\,. \tag{28}$$

(1) $h = -1$

For $h = -1$, there exist only one sub-case.

(1a) $\tau = \alpha \neq 0$

$$a = \frac{\beta}{\gamma}\,, \quad c = -\delta a^3\,. \tag{29}$$

(2) $h = -2$

For $h = -2$, there exist two sub-cases.

(2a) $\tau = 0$, $\alpha = -\frac{1}{2}$

By the same reason, one has

$$-c - 8\delta a^3 - 2\gamma a = 0\,, \quad -7\delta a^2 b - \gamma b + \beta = 0\,,$$

so,

$$b = \frac{\beta}{7\delta a^2 + \gamma}\,, \quad c = -2\gamma a - 8\delta a^3\,. \tag{30}$$

(2b) $\tau = -\frac{1}{2}$, $\alpha = -1$

Similarly, one has

$$-c - 8\delta a^3 = 0\,, \quad -7\delta a^2 b - 2\gamma a = 0\,, \quad -\gamma b + \beta = 0\,,$$

so,

$$a = -\frac{2\gamma^2}{7\delta\beta}\,, \quad b = \frac{\beta}{\gamma}\,, \quad c = -8\delta a^3\,. \tag{31}$$

(3) $h = -3$

For $h = -3$, there exist three sub-cases.

(3a) $\tau = 0$, $\alpha = -\frac{2}{3}$

For this sub-case,

$$c+27\delta a^3+3\gamma a = 0,\ 38\delta a^2 b+2\gamma b = 0,\ 12\delta ab^2-\beta = 0\,.$$

The solutions are

$$a = \pm\left(\frac{-\gamma}{19\delta}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\delta}<0\right),\ b = \mp\left(\frac{\beta}{12\delta a}\right)^{\frac{1}{2}}\left(\frac{\beta}{\delta a}<0\right),$$

$$c = -27\delta a^3 - 3\gamma a. \qquad (32)$$

(3b) $\tau = \alpha = -\frac{1}{3}$

Similarly,

$$c+27\delta a^3 = 0,\ 38\delta a^2 b+3\gamma a-\beta = 0,\ 12\delta ab^2+2\gamma b = 0\,.$$

The solutions are

$$a = -\frac{3\beta}{10\gamma}\,,\quad b = \frac{5\gamma^2}{9\delta\beta}\,,\quad c = -27\delta a^3\,. \qquad (33)$$

(3c) $\tau = -\frac{1}{3}$, $\alpha = -\frac{2}{3}$

The following system is determined by the same reasoning:

$$c+27\delta a^3 = 0,\ 38\delta a^2 b+3\gamma a = 0,\ 12\delta ab^2+2\gamma b-\beta = 0\,,$$

so,

$$a = -\frac{30\gamma^2}{361\delta\beta}\,,\quad b = \frac{19\beta}{20\gamma}\,,\quad c = -27\delta a^3\,. \qquad (34)$$

Besides $h = -1$, $h = -2$ and $h = -3$, there exists only one case, with $h = \frac{1}{\tau}, \alpha = 3\tau$.

(4) $h = \frac{1}{\tau}$, $\alpha = 3\tau$.

It follows that

$$-c+\delta h^3 a^3 = 0\,,\quad \delta a^2 b(h+1)(3h^2+3h+1)+\gamma ha = 0\,,$$

$$3\delta ab^2(h+1)^2(h+2)+\gamma b(h+1) = 0\,,$$

$$\delta b^3(h+1)(h+2)(h+3)+\beta = 0\,.$$

The second and third equations in the above system give $h = \frac{1}{3}$. So, if and only if $h = \frac{1}{3}$ and $\alpha = 3\tau = 9$, parameters a, b and c are given by

$$a = \frac{\gamma}{2}\left(\frac{5}{49\beta\delta^2}\right)^{\frac{1}{3}},\ b = -3\left(\frac{\beta}{280\delta}\right)^{\frac{1}{3}},\ c = \frac{5\gamma^3}{10584\beta\delta}\,. \qquad (35)$$

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