# STABILITY ANALYSIS OF A CERTAIN CLASS OF TIME-VARYING HYBRID DYNAMICAL SYSTEMS

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***Abstract*— In this work we study the exponential stability of a class of hybrid dynamical systems that comprises the sampled-data systems consisting of the interconnection of a time-varying nonlinear continuous-time plant and a time-varying nonlinear discrete-time controller, assuming that the sampling periods are not necessarily constant. For this purpose we develop an Indirect Lyapunov Method of analysis, and show that under adequate hypotheses the exponential stability of the hybrid dynamical system is equivalent to the exponential stability of its linearization.**

***Keywords*— Sampled-data systems; Lyapunov stability; Hybrid systems; Discrete-time systems.**

## I. INTRODUCTION

Sampled-data control systems consisting of the interconnection of a continuous-time nonlinear plant (described by a system of non autonomous first order ordinary differential equations) and a nonlinear digital controller (described by a time-varying system of first order difference equations) is a hybrid system, in the sense that some of its variables evolve smoothly in continuous time while the others change only in a discrete set of time instants. The coexistence of these two different time scales makes it hard to analyze the stability properties of this kind of systems.

Stability analysis of sampled-data systems in the whole time scale was primarily studied for linear systems (Francis and Georgiou, 1988; Iglesias, 1994). The nonlinear case was addressed in the recent papers (Hou *et al.*, 1997; Mancilla-Aguilar *et al.*, 2000; Hu and Michel, 2000a, 2000b), in which qualitative properties of sampled-data control systems, where the plant and the controller are time-invariant, were obtained.

In (Mancilla Aguilar *et al.*, 2000), in particular, we analyzed the stability of a class of hybrid dynamical systems, those described by time-invariant *hybrid equations*, that contains as particular cases the interconnected system consisting of a continuous time plant and a digital controller and that of a continuous time plant and a certain class of hybrid controllers presented in (Rui *et al.*, 1997). In this work we consider the class of hybrid systems described by time-varying hybrid equations with non regular sampling times, and extend some results of (Mancilla Aguilar *et al.*, 2000) to the class of hybrid systems described by this type of equations. To be more precise, we obtain necessary and sufficient conditions for the exponential stability of the system in terms of its linearization, developing in this way an *Indirect Lyapunov Method* for this kind of dynamical systems. Our results can be straightforwardly applied to the study of the exponential stability of sampled-data systems whose plant and controller are nonlinear and time-varying, as these systems are a particular class of those under study. They also justify the method of design of nonlinear digital local exponential stabilizers based on the discretized model of the linearization of the original nonlinear control system (a more complete treatement of these topics can be found in (Mancilla Aguilar, 2001)). Two reasons motivated the treatment of non-regular sampling. The first is the uncertainty that could appear in the frequency of the sampler oscillator, that in certain applications should be of importance. The second is that our results can also be applied to the stability analysis of certain classes of switched systems whose switching times are not necessarily regularly spaced.

The paper is organized as follows. In section II we establish some notation and state the main result of the paper. In section III we present some results about the exponential stability of a perturbed discrete-time time-varying linear system. In section IV we use these results to obtain stability properties of perturbed hybrid linear systems, that enable us to develop the Indirect Lyapunov Method already mentioned, and prove the main result of the paper. Finally, in section V we present some conclusions.

## II. NOTATION AND MAIN RESULT

First, we introduce some notation that will be used in this work.

Let $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}^+$ the sets of non-negative integer, real and non-negative real numbers respectively. We consider the real $q$-space $\mathbb{R}^q$ as a normed space, with norm $|\cdot|$. We denote by $\Omega = \mathbb{R}^n \times \mathbb{R}^m$, its elements $(\xi, z) = \omega$, and we consider in $\Omega$ the norm $\|\omega\| = |\xi| + |z|$. $B_\rho \subset \Omega$ is the closed ball of radius $\rho$ centered at the origin. Given $r > 0$ a function $\varphi : [0, r] \to \mathbb{R}^+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$.

We denote with $\Pi := \{t_k, k \in \mathbb{N}_0\} \subset \mathbb{R}^+$ the set of sampling points and assume that $0 = t_0 < t_1 < \cdots$, $\sup_k(t_{k+1} - t_k) < \infty$ and $\lim_{k\to\infty} t_k = \infty$; $\|\Pi\| = \sup_k(t_{k+1} - t_k)$ is the norm of $\Pi$.

In this paper we will study qualitative properties of the hybrid system $\Sigma_H$ described by the time-varying hybrid equations

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & f_1(t, x(t), x(t_k), z(t_k))\ t_k \le t < t_{k+1} \\ z(t_{k+1}) & = & f_2(k, x(t_k), z(t_k))\ k \in \mathbb{N}_0. \end{array}\right. \tag{1}$$

As was mentioned in the Introduction, this type of systems includes, as a particular case, the sampled-data system consisting of a plant described by the set of time-varying equations

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & f(t, x(t), u(t)) \\ y(t) & = & h(t, x(t)) \end{array}\right. \tag{2}$$

where the state $x(t) \in \mathbb{R}^n$, the control $u(t) \in \mathbb{R}^l$, the output $y(t) \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$, and a digital controller described by the difference equations

$$\left\{\begin{array}{rcl} z(t_{k+1}) & = & g(k, z(t_k), w(t_k)) \\ v(t_k) & = & r(k, z(t_k), w(t_k)) \end{array}\right. \tag{3}$$

with states $z(t_k) \in \mathbb{R}^m$, inputs $w(t_k) \in \mathbb{R}^p$, and outputs $v(t_k) \in \mathbb{R}^l$, interconnected via sampling and zero-order hold (ZOH), i. e. $w(t_k) = y(t_k)$ and $u(t) = v(t_k)$, $t_k \le t < t_{k+1}$ $k \in \mathbb{N}_0$.

In order to assure the existence and uniqueness of the solutions of (1) (see (Mancilla Aguilar *et al.*, 2000; Mancilla Aguilar, 2001) for the definition of solution of (1)) we assume that the function $f_1 : \mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ verifies:

**H1** For each $t \in \mathbb{R}^+$ and $(\xi, z) \in \Omega$, $f_1(t, \cdot, \xi, z)$ is continuous in $\mathbb{R}^n$, and for each $(x, \xi, z) \in \mathbb{R}^n \times \Omega$, $f_1(\cdot, x, \xi, z)$ is Lebesgue measurable in $\mathbb{R}^+$.

**H2** For each compact set $K \subset \mathbb{R}^n$ and each $(\xi, z) \in \Omega$ there exist a constant $L^* \ge 0$ such that $|f_1(t, x, \xi, z) - f_1(t, x', \xi, z)| \le L^*|x - x'|$ $\forall t \in \mathbb{R}^+$ $\forall x, x' \in K$.

We suppose in addition that the origin is an equilibrium for (1), i. e. $f_1(t, 0, 0, 0) = 0$ $\forall t \in \mathbb{R}^+$ and $f_2(k, 0, 0) = 0$ $\forall k \in \mathbb{N}_0$, that $f_1(t, \cdot, \cdot, \cdot)$ is differentiable at $(0, 0, 0)$ for all $t \in \mathbb{R}^+$ and that $f_2(k, \cdot, \cdot)$ is differentiable at $(0, 0)$ for all $k \in \mathbb{N}_0$. Let

$$A(t) = \frac{\partial f_1}{\partial x}(t, 0, 0, 0), B_{11}(t) = \frac{\partial f_1}{\partial \xi}(t, 0, 0, 0),$$

$$B_{12}(t) = \frac{\partial f_1}{\partial z}(t, 0, 0, 0), \tag{4}$$

$$B_{21}(k) = \frac{\partial f_2}{\partial x}(k, 0, 0), B_{22}(k) = \frac{\partial f_2}{\partial z}(k, 0, 0). \tag{5}$$

Then, associated with $\Sigma_H$, we have the linear hybrid system $\Sigma_{LH}$ described by the linearization of (1) at the origin:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B_{11}(t)x(t_k) + B_{12}(t)z(t_k) \\ & & t_k \le t < t_{k+1} \\ z(t_{k+1}) & = & B_{21}(k)x(t_k) + B_{22}(k)z(t_k)\ k \in \mathbb{N}_0 \end{array}\right.$$

Let us recall the definition of local exponential stability of system (1): we say that the origin is an exponentially stable equilibrium of system (1) if there exist positive numbers $\eta$, $\mu$ and $r$ such that for any $\omega = (\xi, z) \in B_r$, and any $t_k \in \Pi$, $\|\phi(t, t_k, \omega)\| \le \eta\|\omega\|e^{-\mu(t-t_k)}$, for all $t \ge t_k$, where $\phi(t, t_k, \omega)$ is the maximally defined solution of (1), starting at $\omega$ in time $t_k$.

The following theorem, which is the main result of this work, establishes stability properties of the origin for $\Sigma_H$ based on its stability properties for $\Sigma_{LH}$. In other words, we will develop an *Indirect Lyapunov Method* for dynamic systems described by equations of the type (1).

**Theorem II.1** Suppose that $A(t)$, $B_{11}(t)$ and $B_{12}(t)$ in (4) are bounded and that the nonlinear terms resulting from the linearization of (1) $g_1(t, x, \xi, z) = f_1(t, x, \xi, z) - A(t)x - B_{11}(t)\xi - B_{12}(t)z$ and $g_2(k, \xi, z) = f_2(k, \xi, z) - B_{21}(k)\xi - B_{22}(k)z$ verify

**H3** $$\lim_{|x|+|\xi|+|z|\to 0} \frac{|g_1(t, x, \xi, z)|}{|x| + |\xi| + |z|} = 0 \quad \text{uniformly in } t \in \mathbb{R}^+$$

and

**H4** $$\lim_{|\xi|+|z|\to 0} \frac{|g_2(k, \xi, z)|}{(t_{k+1} - t_k)(|\xi| + |z|)} = 0 \quad \text{uniformly in}$$

$k \in \mathbb{N}_0$.

Then, the origin is an exponentially stable equilibrium for $\Sigma_H$ if and only if it is an exponentially stable equilibrium for $\Sigma_{LH}$.

**Remark II.1** Hypothesis **H3** and **H4** (without the term $t_{k+1} - t_k$ in the denominator) are standard ones of the First Method of Lyapunov in the stability theory of Ordinary Differential and Difference Equations respectively. The term $t_{k+1} - t_k$ must be included in **H4** because we do not consider that $\delta_k = t_{k+1} - t_k$ is bounded from below by a positive constant. In the case that $\inf \delta_k = 0$, Theorem II.1 is not valid if that term is removed from **H4**.

## III. ON THE EXPONENTIAL STABILITY OF DISCRETE-TIME SYSTEMS

In order to prove Theorem II.1 we need some results about the exponential stability of the discrete-time system described by the equation

$$\omega(t_{k+1}) = H_k(\omega(t_k)) \tag{6}$$

where $H_k : \Omega_k \to \Omega$ and $\Omega_k \subseteq \Omega$ for all $k \in \mathbb{N}_0$.

We denote with $\phi(t_k, t_{k_0}, \omega)$ the solution in time $t_k$

of (6) with the initial condition $(t_{k_0}, \omega)$, and assume that there exists $r > 0$ such that $B_r \subset \Omega_k$ and that $H_k(0) = 0$ for all $k \in \mathbb{N}_0$. It follows from the last assumption that the origin is an equilibrium of (6). The following result, that can be easily proved employing standard techniques of Lyapunov Stability theory, gives a sufficient condition for the exponential stability of the origin of (6) in terms of Lyapunov functions.

**Proposition III.1** Suppose there exists a scalar function $V : \mathbb{N}_0 \times B_r \to \mathbb{R}^+$ such that
1. $c_1\|\omega\|^2 \le V(k, \omega) \le c_2\|\omega\|^2 \ \forall k \in \mathbb{N}_0 \ \forall \omega \in B_r$;
2. $V(k+1, H_k(\omega)) - V(k, \omega) \le -(t_{k+1} - t_k)c_3\|\omega\|^2$ $\forall k \in \mathbb{N}_0 \ \forall \omega \in B_{r'}$,

with $0 < r' \le r$ and, $c_1, c_2$ and $c_3$ positive constants. Then the origin is an exponentially stable (ES) equilibrium of (6), i. e., there exist positive numbers $r^*$, $\mu$ and $\eta$ such that $\|\phi(t_k, t_{k_0}, \omega)\| \le \eta\|\omega\|e^{-\mu(t_k - t_{k_0})} \ \forall t_k \ge t_{k_0}$, $\forall \omega \in B_{r^*}$.

It is possible, when the discrete-time system is *linear*, to obtain propositions that are converse to Proposition III.1, i. e. it is possible to obtain converse Lyapunov theorems. Next, we consider the discrete-time *linear system*

$$\omega(t_{k+1}) = A_k \omega(t_k) \tag{7}$$

with $A_k$ matrices of proper dimensions. Then, $\Omega_k = \Omega$ and 0 will always be an equilibrium for (7).
In addition, $\phi(t_k, t_{k_0}, \omega_0) = \Phi(t_k, t_{k_0})\omega_0$, where $\Phi(t_k, t_{k_0})$ is the *transition matrix*, that verifies

- $\Phi(t_{k_0}, t_{k_0}) = I$ and
- $\Phi(t_k, t_{k_0}) = A_{k-1}A_{k-2}\cdots A_{k_0} \ \ \forall k > k_0$.

**Remark III.1** Due to the linearity of $\phi(t_k, t_{k_0}, \omega_0)$ with respect to $\omega_0$, the following facts are easily established:
1) If the origin is an exponentially stable equilibrium, it is a global one, *i. e.* there exist positive constants $\eta$ and $\mu$ such that $\|\phi(t_k, t_{k_0}, \omega_0)\| \le \eta\|\omega_0\|e^{-\mu(t_k - t_{k_0})}$ for all $k \ge k_0$ and all $\omega_0 \in \mathbb{R}^n$.
2) Due to this last inequality, $\Phi(t_k, t_{k_0})$ verifies $\|\Phi(t_k, t_{k_0})\| \le \eta e^{-\mu(t_k - t_{k_0})}$ for all $k \ge k_0$.

The following converse Lyapunov theorem will be useful in the sequel.

**Theorem III.1** Suppose the origin is an ES equilibrium for (7). Then, there exists a scalar function $V : \mathbb{N}_0 \times \Omega \to \mathbb{R}^+$ that verifies, for all $k \in \mathbb{N}_0$ and all $\omega, \omega' \in \Omega$:
a) $c_1\|\omega\|^2 \le V(k, \omega) \le c_2\|\omega\|^2$;
b) $V(k+1, A_k\omega) - V(k, \omega) \le -(t_{k+1} - t_k)c_3\|\omega\|^2$;
c) $|V(k, \omega) - V(k, \omega')| \le c_4\|\omega + \omega'\|\,\|\omega - \omega'\|$,
with $c_1, \ldots, c_4$ positive constants.

**Proof** Consider the semidefinite positive function defined by

$$V_1(k, \omega) = \omega^T P(k)\omega \tag{8}$$

with $P(k)$ the positive definite matrix given by

$$P(k) = \sum_{j \ge k}(t_{j+1} - t_j)\Phi^T(t_j, t_k)\Phi(t_j, t_k). \tag{9}$$

In order to prove that this matrix is well defined we recall that as the origin is exponentially stable, then $\|\Phi(t_i, t_k)\| \le \eta e^{-\mu(t_i - t_k)}$ (see Remark III.1); hence

$$\begin{aligned}\|P(k)\| &\le \sum_{j \ge k}(t_{j+1} - t_j)\eta^2 e^{-2\mu(t_j - t_k)}\\ &= \sum_{j \ge k}(t_{j+1} - t_j)\eta^2 e^{-2\mu(t_{j+1} - t_k)}e^{2\mu(t_{j+1} - t_j)}\\ &\le \eta^2 e^{2\mu\|\Pi\|}\sum_{j \ge k}(t_{j+1} - t_j)\eta^2 e^{-2\mu(t_{j+1} - t_k)}\\ &\le M,\end{aligned}$$

since the series above is uniformly bounded. In consequence, $0 \le V_1(k, \omega) \le M\|\omega\|^2$. In addition, from (8) and the bound $\|P(k)\| \le M \ \forall k \in \mathbb{N}_0$ we easily obtain

$$|V_1(k, \omega) - V_1(k, \omega')| \le M\|\omega + \omega'\|\|\omega - \omega'\|,$$

for all $\omega, \omega' \in \Omega$.

On the other hand, as $\Phi(t_j, t_{k+1})A_k\omega = \Phi(t_j, t_k)\omega$ for all $j > k \ge 0$ and all $\omega \in \Omega$, it follows that

$$V_1(k+1, A_k\omega) - V_1(k, \omega) = -(t_{k+1} - t_k)\|\omega\|^2.$$

Consider now

$$V_2(k, \omega) = \max_{j \ge k}\omega^T\Phi^T(t_j, t_k)\Phi(t_j, t_k)\omega.$$

This function is well defined since, due to the exponential stability of the origin, $\lim_{j\to\infty}\Phi(t_j, t_k) = 0$ for all $k \ge 0$. In addition, as $\Phi(t_k, t_k) = I$, then $V_2(k, \omega) \ge \|\omega\|^2$ and as $\|\Phi(t_i, t_k)\| \le \eta e^{-\mu(t_i - t_k)}$, we have $V_2(k, \omega) \le \eta^2\|\omega\|^2$.

Let us fix $k \ge 0$; it follows that for a given $\omega \in \Omega$ there exists $j^* = j^*(\omega) \ge k$ such that

$$V_2(k, \omega) = \omega^T\Phi(t_{j^*}, t_k)^T\Phi(t_{j^*}, t_k)\omega.$$

As a consequence, for $\omega$, $\omega' \in \Omega$,

$$\begin{aligned}V_2(k, \omega) - V_2(k, \omega') &\le \omega^T\Phi(t_{j^*}, k)^T\Phi(t_{j^*}, k)\omega\\ &\quad -\omega'^T\Phi(t_{j^*}, k)^T\Phi(t_{j^*}, k)\omega'\\ &\le \eta^2\|\omega + \omega'\|\|\omega - \omega'\|,\end{aligned}$$

with $\eta$ as above. Then, by symmetry, $V_2(k, \omega') - V_2(k, \omega) \le \eta^2\|\omega + \omega'\|\|\omega - \omega'\|$, and we obtain

$$|V_2(k, \omega) - V_2(k, \omega')| \le \eta^2\|\omega + \omega'\|\|\omega - \omega'\|.$$

Recalling once again that $\Phi(t_j, t_{k+1})A_k\omega = \Phi(t_j, t_k)\omega$ for all $j > k \ge 0$ and all $\omega \in \Omega$, we deduce that

$$V_2(k+1, A_k\omega) - V_2(k, \omega) \le 0.$$

Finally, consider $V(k,\omega) = V_1(k,\omega) + V_2(k,\omega)$. It is easy to see that this function verifies the thesis with $c_1 = c_3 = 1$ and $c_2 = c_4 = M + \eta^2$. ∎

The previous results enable us to study the qualitative properties of *perturbations of system* (7), described by an equation of the form

$$\omega(t_{k+1}) = A_k\omega(t_k) + F_k(\omega(t_k)) \tag{10}$$

where $F_k : \Omega_k \to \Omega$ is the *perturbation term*, and $\Omega_k$ are subsets of $\Omega$ such that there exists $r > 0$ with $B_r \subset \Omega_k$ for all $k \in \mathbb{N}_0$. Next we will address the case of *vanishing perturbations* (when $F_k(0) = 0$ for all $k \in \mathbb{N}_0$) and will establish robust stability results in the sense of Lyapunov of the trivial solution of (10).

**Proposition III2** Consider the discrete-time system described by equation (10) and assume that

$$\lim_{\|\omega\|\to 0} \frac{\|F_k(\omega)\|}{(t_{k+1}-t_k)\|\omega\|} = 0 \tag{11}$$

uniformly in $k \in \mathbb{N}_0$. Then, if the origin is an ES equilibrium of (7), it is also an ES equilibrium of (10).

**Proof** Due to the exponential stability of the origin with respect to (7) $\Phi(t_k, t_{k_0}) \leq \eta e^{-\mu(t_k - t_{k_0})} \leq \eta$ for every $k \geq k_0$ and certain positive numbers $\eta, \mu$ (see Remark III1); it follows that $\|A_k\| \leq \eta$ for all $k \in \mathbb{N}_0$. Consider now the scalar function $V : \mathbb{N}_0 \times \Omega \to \mathbb{R}^+$ given by Theorem III1 and pick $\varepsilon < 1$ that verifies

$$0 < \varepsilon < \frac{c_3}{2c_4(2\eta + \|\Pi\|)}.$$

Then, due (11) there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|F_k(\omega)\| \leq \varepsilon(t_{k+1} - t_k)\|\omega\| \leq \|\Pi\|\,\|\omega\|$$

for all $k \in \mathbb{N}_0$ and all $\omega \in B_\delta$. Pick now such an $\omega$; then

$$\begin{aligned}
&V(k+1, A_k\omega + F_k(\omega)) - V(k,\omega) = \\
&= V(k+1, A_k\omega + F_k(\omega)) - V(k+1, A_k\omega) \\
&\quad + V(k+1, A_k\omega) - V(k,\omega) \\
&\leq |V(k+1, A_k\omega + F_k(\omega)) - V(k+1, A_k\omega)| \\
&\quad - (t_{k+1} - t_k)c_3\|\omega\|^2 \\
&\leq \|2A_k\omega + F_k(\omega)\|c_4\|F_k(\omega)\| - (t_{k+1} - t_k)c_3\|\omega\|^2 \\
&\leq (2\eta + \|\Pi\|)c_4\varepsilon\|\omega\|^2(t_{k+1} - t_k) - (t_{k+1} - t_k)c_3\|\omega\|^2 \\
&< -\frac{c_3}{2}(t_{k+1} - t_k)\|\omega\|^2.
\end{aligned}$$

Then, the function $V$ satisfies the hypotheses of Proposition III1 with $\delta$ and $\frac{c_3}{2}$ instead of $r'$ and $c_3$ respectively. It follows that the origin is exponentially stable for the perturbed system (10). ∎

**Remark III2** Examples can be exhibited which show that Proposition III2 fails if condition (11) is replaced by the following weaker one:

$$\lim_{\|\omega\|\to 0} \frac{\|F_k(\omega)\|}{\|\omega\|} = 0, \tag{12}$$

uniformly in $k \in \mathbb{N}_0$.

Nevertheless the converse of Proposition III2 holds under this hypothesis.

**Proposition III3** Consider the discrete-time system described by equation (10) and assume that it verifies (12). Then, if the origin is an ES equilibrium of (10), it is also an ES equilibrium of (7).

## IV. STABILITY OF PERTURBED HYBRID LINEAR SYSTEMS

In this section we apply the results of the previous section to the analysis of the stability properties of the origin of $\Omega = \mathbb{R}^n \times \mathbb{R}^m$ for the perturbed hybrid linear system $\Sigma_{HLP}$ described by the equations

$$\left\{\begin{array}{rcl}
\dot{x}(t) &=& A(t)x(t) + B_{11}(t)x(t_k) + B_{12}(t)z(t_k) \\
&& +g_1(t, x(t), x(t_k), z(t_k)),\ t_k \leq t < t_{k+1} \\
z(t_{k+1}) &=& B_{21}(k)x(t_k) + B_{22}(k)z(t_k) \\
&& +g_2(k, x(t_k), z(t_k)),\ k \in \mathbb{N}_0
\end{array}\right. \tag{13}$$

where $A(\cdot), B_{11}(\cdot)$ and $B_{12}(\cdot)$ are bounded Lebesgue measurable matrix functions in $\mathbb{R}^+$, and $B_{21}(\cdot)$ and $B_{22}(\cdot)$ are matrix functions defined in $\mathbb{N}_0$. We assume that the *perturbation term* $g_1$ verifies **H1** and **H2**. We will also assume that the perturbations are *vanishing*, i. e. $g_1(t,0,0,0) = 0$ and $g_2(k,0,0) = 0$ for all $t \in \mathbb{R}^+$ and $k \in \mathbb{N}_0$ respectively. Using (13) we compute the discretized perturbed hybrid linear system $\Sigma_{DHLP}$

$$\left\{\begin{array}{rcl}
x(t_{k+1}) &=& A_d(k)x(t_k) + B_d(k)z(t_k) \\
&& +\tilde{g}_1(k, x(t_k), z(t_k)), \\
z(t_{k+1}) &=& B_{21}(k)x(t_k) + B_{22}(k)z(t_k) \\
&& +g_2(k, x(t_k), z(t_k)),\ k \in \mathbb{N}_0
\end{array}\right. \tag{14}$$

where
$B_d(k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)B_{12}(s)ds$,
$A_d(k) = \Phi(t_{k+1}, t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)B_{11}(s)ds$,
$\tilde{g}_1(k, x(t_k), z(t_k)) = \int_{t_k}^{t_{k+1}} \Lambda(s, x(s), x(t_k), z(t_k))ds$,
$\Lambda(s, x(s), x(t_k), z(t_k)) = \Phi(t_{k+1}, s)g_1(s, x(s), x(t_k), z(t_k))$ and $\Phi(t,s)$ is the transition matrix corresponding to the matrix equation $\dot{X} = A(t)X,\ X(s) = I$. Let $\omega(t_k) = (x(t_k)^T, z(t_k)^T)^T$; then (14) can be written

$$\begin{array}{rcl}
\omega(t_{k+1}) &=& A_k\omega(t_k) + F_k(\omega(t_k)) \quad (15) \\
\text{where}\quad A_k &=& \begin{bmatrix} A_d(k) & B_d(k) \\ B_{21}(k) & B_{22}(k) \end{bmatrix}, \\
\text{and}\quad F_k(\omega(t_k)) &=& \begin{bmatrix} \tilde{g}_1(k, x(t_k), z(t_k)) \\ g_2(k, x(t_k), z(t_k)) \end{bmatrix}.
\end{array}$$

Let us also consider the unperturbed (*i. e.* with $g_1 \equiv 0$ and $g_2 \equiv 0$) hybrid linear system $\Sigma_{HLU}$ associated with $\Sigma_{HLP}$:

$$\left\{\begin{array}{rl}
\dot{x}(t) &= A(t)x(t) + B_{11}(t)x(t_k) + B_{12}(t)z(t_k) \\
& \quad t_k \leq t < t_{k+1} \\
z(t_{k+1}) &= B_{21}(k)x(t_k) + B_{22}(k)z(t_k)\ k \in \mathbb{N}_0
\end{array}\right.$$

and its discretization, the discretized unperturbed hybrid linear system $\Sigma_{DULS}$, which coincides with the unperturbed discrete hybrid linear system associated with $\Sigma_{DHLP}$:

$$\begin{cases} x(t_{k+1}) & = A_d(k)x(t_k) + B_d(k)z(t_k), \\ z(t_{k+1}) & = B_{21}(k)x(t_k) + B_{22}(k)z(t_k),\ k \in \mathbb{N}_0 \end{cases}$$

Now we may state the main result of this section.

**Theorem IV.1** Suppose that $g_1$ and $g_2$ in (13) verify in addition **H3** and **H4**, respectively.

Then, the following properties are equivalent:
1. The origin is an ES equilibrium for $\Sigma_{HLP}$.
2. The origin is an ES equilibrium for $\Sigma_{HLU}$.
3. The origin is an ES equilibrium for $\Sigma_{DHLP}$.
4. The origin is an ES equilibrium for $\Sigma_{DHLU}$.

**Proof** 1. $\Rightarrow$ 3. and 2. $\Rightarrow$ 4. hold trivially and since 4. $\Rightarrow$ 2. is a particular case of 3. $\Rightarrow$ 1., then it suffices to prove 3. $\Rightarrow$ 1. and 4. $\Leftrightarrow$ 3.

First we show that 3. $\Rightarrow$ 1. In order to prove it we need a technical lemma. Let $\omega = (\xi, z) \in \Omega$; we denote with $\phi(t, t_k, \omega)$ the solution of (13) with initial conditions $(t_k, \omega)$ and with $[t_k, T(k, \omega))$ its maximal interval of definition. We have the following lemma.

**Lemma IV.1** Asumme that the hypotheses of Theorem IV.1 hold. Then there exist positive constants $c$ and $r^*$ such that for all $k \in \mathbb{N}_0$ and all $\omega \in B_{r^*}$, $\|\phi(t, t_k, \omega)\| \leq c\|\omega\|\ \forall t \in [t_k, t_{k+1})$.

**Proof** Since $g_1$ satisfies **H3**, there exist $r > 0$ and $\rho : [0, r] \to \mathbb{R}^+$ of class $\mathcal{K}$ such that $|g_1(t, x, \xi, z)| \leq \rho(|x| + |\xi| + |z|)(|x| + |\xi| + |z|)$ for all $t \in \mathbb{R}^+$ and all $(x, \xi, z) \in \mathbb{R}^n \times \Omega : |x| + |\xi| + |z| \leq r$.

Take $\mu : 0 < \mu < r$ and $r^* = \frac{\mu}{2(1+\delta)}$ with

$$\delta = (1 + \rho(r)\|\Pi\| + \zeta\|\Pi\|)e^{\rho(r)\|\Pi\| + \zeta\|\Pi\|},$$

and $\zeta \geq 0$ such that $\|A(t)\| + \|B_{11}(t)\| + \|B_{12}(t)\| \leq \zeta$ $\forall t \in \mathbb{R}^+$, and consider $\omega = (\xi, z) \in B_r^*$. Then, for all $t \in [t_k, t^*) = [t_k, T(k, \omega)) \cap [t_k, t_{k+1})$, $\phi(t, t_k, \omega) = (x(t), z)$ with $x(t) = \xi + \int_{t_k}^t [A(s)x(s) + B_{11}(s)\xi + B_{12}(s)z + g_1(s, x(s), \xi, z)]ds$.
Let $\mathcal{I} = \{t \in [t_k, t^*) : |x(s)| < \frac{\mu}{2}\ \forall s \in [t_k, t]\}$; then, $t_k \in \mathcal{I}$ since $|x(t_k)| = |\xi| \leq r^* < \frac{\mu}{2}$. In consequence $\mathcal{I} \neq \emptyset$ and it follows from the continuity of $x(\cdot)$ that $\mathcal{I} = [t_k, t^{**})$ with $t^{**} \leq t^*$.
Consider now $t \in \mathcal{I}$, then $|x(t)| + |\xi| + |z| < \frac{\mu}{2} + r^* < \mu < r$ and in consequence, for all $t \in \mathcal{I}$,

$$\begin{aligned}
&|x(t)| \leq |\xi| \\
&+ \int_{t_k}^t \|A(s)\||x(s)| + \|B_{11}(s)\||\xi| + \|B_{12}(s)\||z|ds \\
&+ \int_{t_k}^t \rho(|x(s)| + |\xi| + |z|)(|x(s)| + |\xi| + |z|)ds \\
&\leq |\xi| + (|\xi| + |z|)\int_{t_k}^t (\zeta + \rho(r))ds \\
&+ \int_{t_k}^t (\zeta + \rho(r))|x(s)|ds \\
&\leq |\xi| + (|\xi| + |z|)(\zeta\|\Pi\| + \rho(r)\|\Pi\|) \\
&+ \int_{t_k}^t (\zeta + \rho(r))|x(s)|ds \\
&< (|\xi| + |z|)(1 + \zeta\|\Pi\| + \rho(r)\|\Pi\|) \\
&+ \int_{t_k}^t (\zeta + \rho(r))|x(s)|ds \\
&\leq (1 + \rho(r)\|\Pi\| + \zeta\|\Pi\|)e^{\rho(r)\|\Pi\| + \zeta\|\Pi\|}(|\xi| + |z|) \\
&= \delta(|\xi| + |z|),
\end{aligned}$$

where the last inequality is obtained applying Gronwall's Lemma. It follows that $t^{**} = t^*$; (if $t^{**} < t^*$, then due to the continuity of $x(\cdot)$, $|x(t^{**})| = \lim_{t \to t^{**}-} |x(t)| \leq \delta(|\xi| + |z|) \leq \delta r^* < \frac{\mu}{2}$ and $t^{**} \in \mathcal{I}$ which is a contradiction). In consequence, for all $t \in [t_k, t^*)$, and all $\omega = (\xi, z) \in B_{r^*}$, $\|\phi(t, t_k, \omega)\| = |x(t)| + |\xi| + |z| \leq \delta(|\xi| + |z|) + |\xi| + |z| = (\delta + 1)(|\xi| + |z|) = c\|\omega\|$. Finally we prove that $t^* = t_{k+1}$; be this not the case $t^* = T(k, \omega)$. But $\phi(t, t_k, \omega) \in B_c$ for all $t \in \mathcal{I}$ and, due to standard theorems about ordinary differential equations, $t^* < T(k, \omega)$ which is a contradiction, and the lemma follows. ∎

Now we prove 3. $\Rightarrow$ 1. Suppose then that 0 is an ES equilibrium of $\Sigma_{DHLP}$ and let $\phi^*(t_k, t_{k_0}, \omega)$ the solution of (14) with initial conditions $(t_{k_0}, \omega)$. Then there exist positive numbers $\eta^*, \mu^*$ and $\kappa^*$ such that $\forall k \geq k_0 \in \mathbb{N}_0\ \forall \omega \in B_{\kappa^*}$,

$$\|\phi^*(t_k, t_{k_0}, \omega)\| \leq \eta^*\|\omega\|e^{-\mu^*(t_k - t_{k_0})}.$$

Let $r^*$ and $c$ as in Lemma IV.1 and $\kappa = \min\{\frac{r^*}{\eta^*}, \kappa^*\}$; hence $\forall k \geq k_0 \in \mathbb{N}_0\ \forall \omega, \|\omega\| < \kappa$,

$$\|\phi^*(t_k, t_{k_0}, \omega)\| \leq \eta^*\|\omega\|e^{-\mu^*(t_k - t_{k_0})}.$$

Consider $\omega \in B_\kappa$ and $t$ such that $t_{k_0} \leq t_k \leq t < t_{k+1}$ for $k \geq k_0 \in \mathbb{N}_0$. Then, by Lemma IV.1,

$$\begin{aligned}
\|\phi(t, t_{k_0}, \omega)\| &\leq c\|\phi^*(t_k, t_{k_0}, \omega)\| \\
&\leq c\eta^*\|\omega\|e^{-\mu^*(t_k - t_{k_0})} \\
&= c\eta^*\|\omega\|e^{\mu^*(t - t_k)}e^{-\mu^*(t - t_{k_0})} \\
&\leq c\eta^*\|\omega\|e^{\mu^*\|\Pi\|}e^{-\mu^*(t - t_{k_0})} \\
&= \eta\|\omega\|e^{-\mu^*(t - t_{k_0})},
\end{aligned}$$

and the origin is an ES equilibrium for $\Sigma_{HLP}$.

Next we prove the statement 3. $\Leftrightarrow$ 4. Let $\omega = (\xi, z) \in B_{r^*}$ with $r^*$ as in Lemma IV.1; first we prove that $\tilde{g}_1(k, \xi, z)$ in (14) verifies $\lim_{\|\omega\| \to 0} \frac{\tilde{g}_1(k,\xi,z)}{(t_{k+1}-t_k)\|\omega\|} = 0$ uniformly in $k \in \mathbb{N}_0$. If $x(t)$ is the solution of (13) with initial conditions $(t_k, \omega)$, then according to Lemma IV.1, $|x(t)| \leq \delta(|\xi|+|z|)$ and $(|x(t)|+|\xi|+|z|) < \mu < r$ for all $t \in [t_k, t_{k+1})$, for all $k \in \mathbb{N}_0$. Then, from (13) - (14), we obtain

$$
\begin{aligned}
|\tilde{g}_1(k, \xi, z)| &\leq \int_{t_k}^{t_{k+1}} \|\Phi(t_{k+1}, s)\| |g_1(s, x(s), \xi, z)| ds \\
&\leq \int_{t_k}^{t_{k+1}} e^{\zeta(t_{k+1}-s)} \rho(|x(s)| + |\xi| + |z|) \times \\
&\qquad \times (|x(s)| + |\xi| + |z|) ds \\
&\leq e^{\zeta\|\Pi\|} \int_{t_k}^{t_{k+1}} \rho((\delta+1)(|\xi| + |z|)(\delta+1)(|\xi| + |z|) ds,
\end{aligned}
$$

since, $\|\Phi(t,s)\| \leq e^{\zeta|t-s|}$ with $\zeta$ as in the proof of Lemma IV.1. If we define $\tilde{\rho} : [0, \frac{r}{\delta+1}] \to \mathbb{R}^+$ as $\tilde{\rho}(\lambda) = e^{\zeta\|\Pi\|}(\delta+1)\rho((\delta+1)\lambda)$, $\tilde{\rho}$ is a function of class $\mathcal{K}$ and it follows, by the inequalities above, that $|\tilde{g}_1(k, \xi, z)| \leq (t_{k+1} - t_k)\tilde{\rho}(|\xi| + |z|)(|\xi| + |z|)$, and in consequence, $\lim_{\|\omega\| \to 0} \frac{\tilde{g}_1(k,\xi,z)}{(t_{k+1}-t_k)\|\omega\|} = 0$ uniformly in $k \in \mathbb{N}_0$.

Then, due to the assumption about $g_2(k, \xi, z)$, $F_k(\omega)$ in (15) verifies $\lim_{\|\omega\| \to 0} \frac{F_k(\omega)}{(t_{k+1}-t_k)\|\omega\|} = 0$ uniformly in $k \in \mathbb{N}_0$. Hence, according to Propositions III2 and III3, the origin is ES for $\Sigma_{DHLP}$ if and only if it is ES for $\Sigma_{DHLU}$. ∎

**Remark IV.1** Part 1. $\Leftrightarrow$ 3. of Theorem IV.1 extends the results obtained by Iglesias (1994) for time-varying linear systems in two ways: we consider time-varying sampling periods instead of constant ones and we consider time-varying nonlinear equations.

In addition, some sufficient conditions for the uniform asymptotic stability reported in the literature may be readily obtained from this theorem. In fact, Theorems 1 of (Hou *et al.*,1997), Theorem 1 of (Rui *et al.*, 1997) and Theorem 2.1 of (Hu and Michel, 2000b) are corollaries of Theorem IV.1, since the conditions under which their results hold imply the exponential stability of $\Sigma_{DHLU}$.

**Proof of Theorem II.1** $A(t)$, $B_{11}(t)$ and $B_{12}(t)$ in (4) are bounded (by hypothesis), and are also Lebesgue measurable since $f_1$ in (1) verifies **H1** and **H2**; it follows that $g_1$ and $g_2$ also verify **H1** and **H2**. In addition, since (1) can be written in the form (13), the hypotheses of Theorem IV.1 hold for $\Sigma_H$ and $\Sigma_{LH}$ instead of $\Sigma_{HLP}$ and $\Sigma_{HLU}$ respectively. In consequence, the theorem follows from Theorem IV.1. ∎

## V. Conclusions

In this work we presented results about the exponential stability of the class of hybrid systems described by time-varying hybrid equations, which comprises the class of sampled-data systems consisting of the interconnection of a time-varying nonlinear continuous time plant and a time-varying nonlinear discrete-time controller, assuming that the sampling periods are not necessarily constant. For this purpose we developed an Indirect Lyapunov Method of analysis, and showed that under adequate hypotheses the exponential stability of the hybrid dynamical system is equivalent to the exponential stability of its linearization.

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