QUANTIZED-STATE CONTROL: A METHOD FOR DISCRETE EVENT CONTROL OF CONTINUOUS SYSTEMS

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***Abstract*— This paper introduces a new method for the digital implementation of controllers designed in continuous time. Through the quantization of its state and input variables the original continuous controller is mapped into a discrete event model within the DEVS formalism framework that can be implemented in a digital device. Under certain conditions on the original continuous control system (CCS), this implementation guarantees regional convergence in finite time of system trajectories to arbitrarily small regions around the equilibrium points even in the presence of A/D and D/A quantization effects. The convergence of the new scheme to the CCS is demonstrated when the quantization width goes to zero. Further, a design algorithm for the digital controller is given, which fulfills specifications of admissible final error and convergence speed. Also discussed is the computational efficiency of the scheme, along practical implementation issues. Two numerical examples are provided illustrating some benefits of the new method.**

***Keywords*— Digital Control, Discrete Event Systems, Quantized Systems, Nonlinear Systems.**

I. INTRODUCTION

Practical implementations of most control systems require the use of digital devices. Due to the different nature of the signals present at the inputs and outputs of the digital controller and the continuous-time plant, the interconnection between them must be made through A/D and D/A converters. Since the number of bits used by these converters is finite, a quantization effect takes place over the related variables causing undesired consequences on the stability, steady state error and the general performance of the system.

Because of quantization problems, attracting sets must be considered instead of equilibrium points, and *ultimate boundedness* of solutions instead of asymptotic stability. As shown in (Miller *et al.*, 1988) a SISO linear continuous-time plant with a linear digital controller designed so that the closed loop system is uniformly asymptotically stable, a quantization size exists that transforms the asymptotic stability into ultimate boundedness with arbitrary small bounds. Similar studies were performed in (Farrel and Michel, 1989) with conclusions about the error introduced by the quantization in finite time. These results were extended to nonlinear plants (but with linear controllers) in (Hou *et al.*, 1997) and then, in (Hu and Michel, 1999) a similar analysis is applied to the multirate case. In both cases only local results are obtained since Lyapunov's first method is used to analyze stability properties. Nevertheless, the unattainability of global results is not only due to the quantization, but also to the use of fixed time-discretization. Indeed, a fixed sampling time which is adequate when the state is weakly perturbed away from an equilibrium point could provoke instability for larger state perturbations.

Instead of studying the effects of the quantization after the controller is designed, some works attempt to deal with the quantization at the design stage. In (Delchamps, 1990) the problem of stabilizing a discrete time linear system taking into account the quantization in the state measurement is studied. In (Brockett and Liberzon, 2000; Liberzon, 2000) there is also an study over CCS and the problem of the imposibility of convergence to the equilibrium points is solved by allowing the quantizers to change the size of the quantization intervals. Although the scheme and the problem are completely different, the study on nonlinear systems made there is quite similar to the stability analysis showed in this work.

Since recently, quantization of variables is being applied for simulation purposes. In (Zeigler and Lee, 1998) the authors proposed that continuous time systems can be simulated through the quantization of state and input variables instead of the discretization of time. They also showed that the resulting system can be described by a discrete event model within the DEVS formalism (Zeigler *et al.*, 2000).

This idea was taken in (Kofman and Junco, 2001), where the authors introduce the concept of Quantized State System (QSS), that are continuous time systems where the state variables are quantized through quantization functions with hysteresis[1]. It has been shown that QSS with piecewise constant input trajectories can be exactly represented by DEVS models. Thus, the addition of the mentioned quantization functions to a continuous model transforms it into a QSS that can be simulated in a digital device.

In this paper, based on the idea of QSS, we develop a new digital control scheme called Quantized State Control (QSC). Here the controller is a discrete event system that can be obtained from a continuous (and possibly nonlinear) previously designed controller. The mentioned discrete event model is the representation of a QSS obtained through the quantization with hysteresis of the state variables of the continuous controller. The A/D converters, working in an asynchronous way –which results in an important reduction of the computational costs of the implementation– perform the necessary quantization of the input without introducing time discretization.

Based on Lyapunov's second method, a stability study is made over the QSC scheme giving sufficient conditions to assure regional convergence of the trajectories to small regions around the equilibrium points. When the CCS satisfies some additional conditions, semiglobal convergence can be also achieved. It is important to mention that these stability properties are deduced for the general case of a nonlinear MIMO plant with a nonlinear controller. The absence of time discretization in the scheme is the main key in the proof of the mentioned properties.

The paper is organized as follows. In Section II the concept of Quantized State Systems is introduced and some properties of this class of systems are mentioned. In Section III the QSC scheme is formally defined. The study of stability and convergence is made in Section IV, and based on the stability theorem deduced there, an algorithm that allows the design of the quantized state controller achieving stability and steady state error goals is developed and then illustrated with an example.

Finally, in Section V the reduction of the computational costs is treated through the analysis over a simple system.

II. QUANTIZED STATE SYSTEMS

Quantized State Systems (QSS) are continuous time systems where each state variable is affected by a quantization function equipped with hysteresis.

Before giving a formal definition of QSS, the concept of *quantization function with hysteresis* will be introduced.

[1] The presence of hysteresis in the quantization is necessary to assure that the resulting DEVS models are *legitimate* (Kofman and Junco, 2001). A DEVS model is said to be *legitimate* when it cannot produce an infinite number of events in a finite interval of time (Zeigler *et al.*, 2000)

A. Quantization Functions

Let $D = \{d_0, d_1, ..., d_r\}$ be a set of real numbers where $d_{i-1} < d_i$ with $1 \le i \le r$ and let $x \in \Omega$ be a continuous trajectory, where $x : \mathbb{R} \to \mathbb{R}$. Let $b : \Omega \times t_0 \to \Omega$ be a mapping and let $q = b(x, t_0)$ where the trajectory q for $t \ge t_0$ satisfies

$$q(t) = \begin{cases} d_m & \text{if } t = t_0 \\ d_{i+1} & \text{if } x(t) = d_{i+1} \wedge q(t^-) = d_i \wedge i < r \\ d_{i-1} & \text{if } x(t) = d_i - \varepsilon \wedge q(t^-) = d_i \wedge i > 0 \\ q(t^-) & \text{otherwise} \end{cases} \tag{1}$$

and

$$m = \begin{cases} 0 & \text{if } x(t_0) < d_0 \\ r & \text{if } x(t_0) \ge d_r \\ j & \text{if } d_j \le x(t_0) < d_{j+1} \end{cases}$$

Then, the map b is a *Quantization Function with Hysteresis*. The width of the hysteresis window is ε. The values d_0 and d_r are the lower and upper saturation values. Figure 1 shows a typical quantization function with uniform quantization intervals. A fundamental

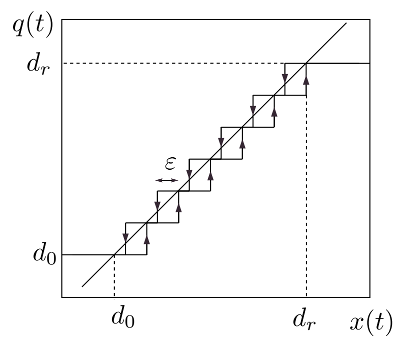

Figure 1: Quantization Function with hysteresis

property of a Quantization Function with hysteresis when $t \ge t_0$ is given by the following inequality:

$$d_0 \le x \le d_r \Rightarrow |q(t) - x(t)| \le \max_{1 \le i \le r}(d_i - d_{i-1}, \varepsilon) \tag{2}$$

B. QSS related to a State Equation System

Consider the State Equation System given by:

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{cases} \tag{3}$$

Related to this system, an associated QSS is defined as follows:

$$\begin{cases} \dot{x}(t) &= f(q(t), u(t)) \\ y(t) &= g(q(t), u(t)) \end{cases} \tag{4}$$

where $q(t)$ and $x(t)$ are related (componentwise) by quantization functions with hysteresis. The components of the vector $q(t)$ are called *quantized variables*. Figure 2 shows a block diagram of a QSS.

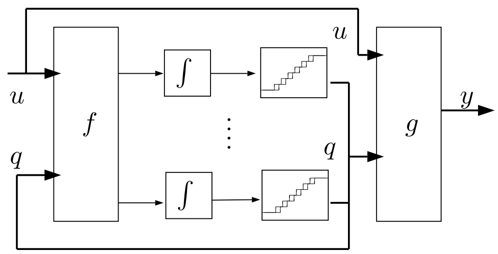

Figure 2: Block Diagram of the QSS

C. Some properties of QSS

The most significant properties of the QSS are related to the form of the trajectories. Provided that the inputs have piecewise constant trajectories and the function f is continuous and bounded in any bounded domain, it can be assured that the quantized variables and the state variable derivatives have piecewise constant trajectories while the state variables have continuous piecewise linear trajectories. Figure 3 shows typical trajectories in a QSS.

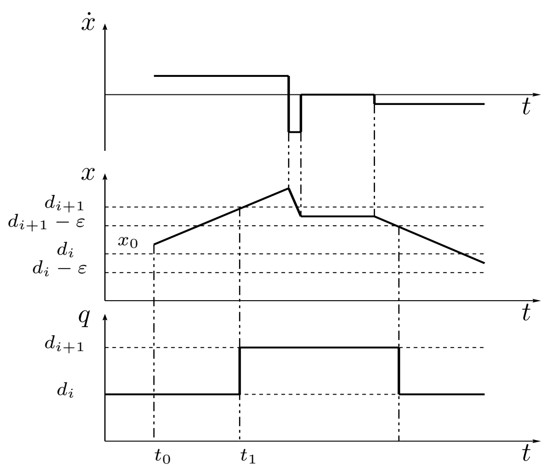

Figure 3: Typical trajectories in a QSS

Because of these properties, representing the piecewise constant trajectories by events allows the exact simulation of QSS by discrete event models within the DEVS formalism framework. The DEVS model related to a generic QSS and the proof of the mentioned properties can be found in (Kofman and Junco, 2001).

The possibility of representing a QSS by a DEVS model and the fact that DEVS models can be simulated in real time by a digital device[2] (Zeigler and Kim, 1993) suggest the use of QSS as digital controllers.

III. QUANTIZED STATE CONTROL

Consider the CCS consisting of plant and controller, Eqs. (5) and (6) respectively, and their (ideal) interconnection, Eq. (7).

$$\begin{cases} \dot{x}_p(t) = f_p(x_p(t), u_p(t)) \\ y_p(t) = g_p(x_p(t)) \end{cases} \tag{5}$$

$$\begin{cases} \dot{x}_c(t) = f_c(x_c(t), u_c(t)) \\ y_c(t) = g_c(x_c(t), u_c(t)) \end{cases} \tag{6}$$

$$u_p(t) = y_c(t), \quad u_c(t) = y_p(t) \tag{7}$$

With this representation, general problems of regulation with linear and nonlinear plants and controllers can be treated.

Definition 1. *The QSS associated to a continuous controller* (6) *is called Quantized State Controller (QSC controller).*

The connections between the plant outputs and the controller inputs require the use of A/D and D/A converters. Since QSC controllers avoid time discretization, it would be desirable that the converters do the same. Thus, the A/D conversions will be performed only when the analog input and the digital output of the converters differ in a quantity corresponding to a quantization interval[3]. Similarly, the D/A conversions will be only performed when the digital outputs of the controller change.

Definition 2. *A QSC system is defined as a control scheme composed by a continuous plant and a QSC controller connected through asynchronous A/D and D/A converters.*

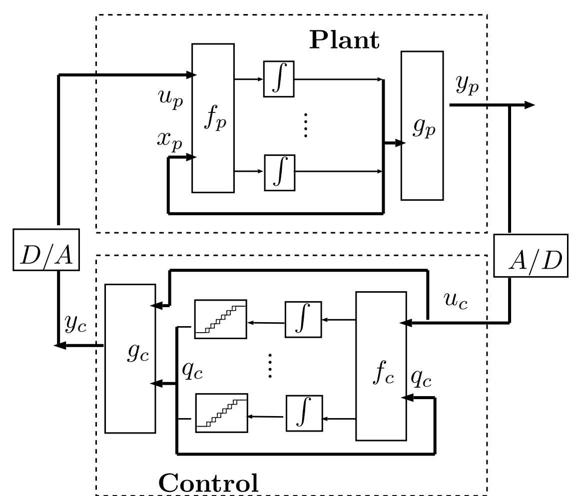

Figure 4: Block Diagram of the QSC system

Figure 4 shows a block diagram representation of a QSC system. Here, according to (5), there is a strictly

[2]DEVS representation of QSS is exact. However, real time simulation of DEVS has errors related to the temporal resolution and the round-off introduced by the digital device

[3]This is an asynchronous sampling technique that is used in some works to obtain fast A/D conversions with a good tradeoff between speed and resolution (Sayiner *et al.*, 1993).

proper plant[4].

The QSC implementation of the controller transforms (6) into the new set of equations:

$$\begin{cases} \dot{x}_c(t) &= f_c(q_c(t), u_c(t)) \\ y_c(t) &= g_c(q_c(t), u_c(t)) \end{cases} \tag{8}$$

where the difference between q_c and x_c is bounded according to (2).

Taking into account the way they work, the asynchronous A/D converters can be seen as quantization functions with hysteresis where the quantization intervals and the hysteresis windows have the same size. In a similar way, the D/A converters –which are also asynchronous but memoryless– can be represented by quantization functions without hysteresis ($\varepsilon = 0$). Thus, the presence of the asynchronous A/D and D/A converters transforms (7) into:

$$u_p(t) = y_{c_q}(t), \;\; u_c(t) = y_{p_q}(t) \tag{9}$$

where the variables $y_{c_q}(t)$ and $y_{p_q}(t)$ are the quantized versions of the plant and the controller output variables, which differ from the continuous y_c and y_p in a quantity bounded by (2).

The presence of quantization with hysteresis in the A/D converters also guarantees that the controller inputs have piecewise constant trajectories. Since the QSC controller is a QSS, it can be exactly represented by a DEVS model and implemented in a digital device.

IV. STABILITY AND CONVERGENCE

One of the most important properties of QSC is the conservation of the stability properties of the original CCS even in presence of A/D and D/A quantization effects. The substitution of time-discretization by the quantization of variables is the key element yielding this property.

A. A stability theorem for QSC

The following theorem shows that, when the CCS has a regional stable equilibrium point a quantization exists such that the QSC system also assures regional convergence of the trajectories to arbitrary regions around that point.

Theorem 1. *Consider that the origin of the closed loop CCS* (5)-(7) *is a regionally stable equilibrium point. Suppose that the functions f_p, g_p, f_c and g_c are continuously differentiable. Further assume that a Lyapunov function V is known, defined in an open region D containing the origin. Then, a QSC system associated to the original CCS can be found, such that all initial conditions lying in an arbitrary interior region of D are attracted in finite time to another arbitrary interior region of the former one. Both interior regions must be limited by level surfaces of V.*

[4]It is also possible to use QSC with plants having relative degree equal to 0. However, in this case, the relative degree of the continuous controller must be at least one to avoid an infinite number of conversions in a finite interval of time

Proof. From equations (5) and (7) the following closed loop equations of the continuous system can be obtained:

$$\begin{cases} \dot{x}_p &= f_p(x_p, g_c(x_c, g_p(x_p))) \\ \dot{x}_c &= f_c(x_c, g_p(x_p)) \end{cases} \tag{10}$$

The implementation of the corresponding QSC system –Eqs. (8) and (7)– transforms (10) into:

$$\begin{cases} \dot{x}_p = f_p(x_p, g_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) + \Delta y_c) \\ \dot{x}_c = f_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) \end{cases} \tag{11}$$

where $\Delta x_c = q_c - x_c$, $\Delta y_p = u_c - y_p$ and $\Delta y_c = u_p - y_c$. Define:

$$\begin{aligned} &\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) = \\ &\quad \frac{\partial V}{\partial x_p}(x) \cdot f_p(x_p, g_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) + \Delta y_c) + \\ &\quad \frac{\partial V}{\partial x_c}(x) \cdot f_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) \end{aligned} \tag{12}$$

with:

$$\begin{aligned} \frac{\partial V}{\partial x_p}(x) &= \left[\frac{\partial V}{\partial x_{p_1}} \cdots \frac{\partial V}{\partial x_{p_n}}\right](x) \\ \frac{\partial V}{\partial x_c}(x) &= \left[\frac{\partial V}{\partial x_{c_1}} \cdots \frac{\partial V}{\partial x_{c_k}}\right](x) \end{aligned}$$

with n and k being the order of the plant and the controller respectively and $V(x) = V(x_p, x_c)$ is the Lyapunov function of the closed loop system defined in (10). From equation (12) it can be easily seen that:

$$\alpha(x, 0, 0, 0) = \dot{V}(x)\Big|_{(10)} \tag{13}$$

Let D_1 be an interior region of D ($D \subset \mathbb{R}^{n+k}$) limited by some level surface of V. Let D_2 be an interior region of D_1 also limited by a level surface of V. Let D_3 be the region defined by $D_3 = D_1 - D_2$.

Since $\dot{V}(x)$ is negative definite, it is possible to find a positive number s so that:

$$\dot{V}(x) < -s, \forall x \in D_3 \tag{14}$$

Let α_M be the function defined by:

$$\alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) = \sup_{x \in D_3} (\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c)) \tag{15}$$

From (13) and (14), it follows that:

$$\alpha_M(0, 0, 0) < -s \tag{16}$$

Since the function α is continuous, the function α_M is also continuous. From this property and (16), given an arbitrary number s_1 ($s > s_1 > 0$), it is possible to find a positive number ρ so that the condition:

$$\|(\Delta x_c, \Delta y_p, \Delta y_c)\| < \rho \tag{17}$$

implies that:

$$\alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) < -s_1 \tag{18}$$

Taking into account (2), the condition given in (17) can be satisfied with the choice of an adequate quantization.[5] Observe that saturation must be outside of region D_1.

Let $x(t)$ be a solution of equation (11) for the initial condition $x(t=0) = x_0 \in D_3$. Consider that the quantization was done in order to satisfy the condition given by (17). From (11) and (12) it follows that:

$$\begin{aligned}\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) &= \frac{\partial V}{\partial x_p}(x) \cdot \dot{x_p} + \frac{\partial V}{\partial x_c}(x) \cdot \dot{x_c} = \\ &= \frac{\partial V}{\partial x}(x) \cdot \dot{x}\end{aligned}$$

Using (15) and (18) in the equation above, we have:

$$\frac{\partial V}{\partial x}(x) \cdot \dot{x} < -s_1 \tag{19}$$

This condition will be satisfied at least during certain time while $x(t)$ remains inside D_3 (this is guaranteed by the continuity of $x(t)$). After integrating both sides of the inequality (19), we have:

$$\begin{aligned}\int_0^t \frac{\partial V}{\partial x}(x) \cdot \dot{x} \cdot dt &< \int_0^t -s_1 \cdot dt \\ V(x(t)) - V(x(0)) &< -s_1 \cdot t \\ V(x(t)) &< V(x_0) - s_1 \cdot t\end{aligned}$$

This implies that V evaluated along the solution is bounded by a strictly decreasing function while that solution remains inside D_3. Since the value $V(x_0)$ is smaller than the value that V takes in the bound of D_1, it is clear that the trajectory will never leave D_1.

Let V_1 be the value that V takes in the bound of region D_2. Then, it can be easily seen that the trajectory will reach the region D_2 in a finite time t_1 with:

$$t_1 < \frac{V(x_0) - V_1}{s_1}$$

which completes the proof. □

Corollary 1. *When the Lyapunov function derivative is negative definite in all the state space, the QSC implementation can assure semiglobal ultimately boundedness.*

[5] For instance, considering the same uniform quantization for all the quantized variables the mentioned condition can be achieved by taking:

$$\max(\Delta q, \varepsilon) < \frac{\rho}{\sqrt{k+m+p}}$$

where Δq and ε are the quantization interval and the hysteresis window size respectively, k is the controller order (i.e. is the size of Δx_c), p is the number of output variables of the plant (size of Δy_p) and m is the number of input variables of the plant (size of Δy_c).

The proof of this corollary is straightforward. Achieving semiglobal stability requires enlarging the region D_1. Unfortunately, it also implies enlarging the saturation region and then, global stabilization cannot be assured in general cases.

B. Convergence of the QSC scheme

It was shown that the QSC implementation can approximate the stability properties of the original controller designed in continuous time. The following theorem shows that the trajectories of the QSC system go to the trajectories of the CCS system when the quantization goes to zero. Thus, any performance measure achieved by the original continuous controller can be approximately accomplished by the QSC controller with the choice of sufficiently small quantization intervals.

Theorem 2. *Consider the CCS* (10) *and the associated QSC implementation* (11)*. Let D_{x_c}, D_{y_c} and D_{y_p} be the non-saturation regions of the QSC controller, the D/A converters and the A/D converters respectively. D_{x_c} is defined as $D_{x_c} = \{x = (x_1, ..., x_k)/d_{0_i} < x_i < d_{r_i}\}$, while D_{y_c} and D_{y_p} are defined in a similar way. Let D_{x_p} be a bounded region in $\mathbb{R}^n$ and let D be a non-saturation region of the QSC system defined by*

$$D = \{(x_p, x_c) | x_p \in D_{x_p}, x_c \in D_{x_c}, \\ g_p(x_p) \in D_{y_p}, g_c(x_c, g_p(x_p)) \in D_{y_c}\}$$

Assume that the functions f_c and g_c are Lipschitz on $D_{x_c} \times D_{y_p}$, the function f_p is Lipschitz on $D_{x_p} \times D_{y_c}$ and the function g_p is Lipschitz on D_{x_p}. Let $\phi(t)$ be the solution of (10) *from the initial condition $x(0) = x_0$ and let $\phi_1(t)$ be a solution of* (11) *starting in the same initial condition x_0. Assume that $\phi(t) \in D_1\ \forall t$ being D_1 a closed interior region of D. Then, $\phi_1(t) \to \phi(t)$ when the quantization intervals go to* 0.

The proof can be found in (Kofman, 2001).

C. A QSC implementation procedure

Based on Theorem 1 the following procedure can be given in order to design QSC controllers satisfying some conditions on convergence.

1. Design a continuous controller that allows finding an appropriate Lyapunov function for the closed loop system.
2. Identify the region D where the derivative of the Lyapunov function is negative.
3. Define the region $D_1 \subset D$ from which the trajectories should converge.
4. Choose $D_2 \subset D_1$ according to the desired steady state error (D_1 and D_2 must be limited by level surfaces of the Lapunov function).
5. Calculate the function α using Eq. (12).
6. Obtain the function α_M with Eq. (16).
7. Find the maximum s satisfying (14) and choose the lower bound of convergence speed s_1 so that $0 < s_1 < s$.

8. Estimate the maximum value of ρ so that (17) implies (18).
9. Choose the quantization intervals in order to satisfy (17).
10. Choose the saturation bounds of the quantization functions outside the region D_1

It can be seen that this procedure for the choice of the quantization intervals guarantees the convergence of the system trajectories to the region D_2.

D. An example of QSC design

The following example illustrates the use of the procedure developed in the previous section.

Consider the plant:

$$\begin{cases} \dot{x}_p(t) & = & x_p^2 + u_p \\ y_p(t) & = & x_p(t) \end{cases} \tag{20}$$

We will suppose that the goal is stabilizing the plant around the origin. The first step in the algorithm is the design of a continuous controller. For instance, the following controller can achieve the mentioned goal.

$$\begin{cases} \dot{x}_c(t) & = & -x_c - u_c \\ y_c(t) & = & x_c(t) - u_c(t) - u_c^2(t) \end{cases} \tag{21}$$

The resulting closed loop equations are:

$$\begin{cases} \dot{x}_p(t) & = & -x_p + x_c \\ \dot{x}_c(t) & = & -x_p - x_c \end{cases}$$

It can be easily verified that the origin is the equilibrium point, and it is asymptotically and globally stable. By taking the Lyapunov function

$$V(x) = \frac{1}{2}x_p^2 + \frac{1}{2}x_c^2 \tag{22}$$

it follows that:

$$\dot{V}(x) = -x_p^2 - x_c^2$$

Since the stability is global ($D = \mathbb{R}^2$) the definition of the region D_1 will be only neccessary for the choice of the saturation bounds. Suposse also that the goal is assuring the convergence of the trajectories to the region $D_2 = \{x/\|x\| < 1\}$. ($\|\cdot\|$ stands for the euclidean norm).

It follows from (12), (20), (21) and (22) that:

$$\begin{aligned} &\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) = \\ &\quad -x_p^2 - x_c^2 + x_p(\Delta x_c - \Delta y_p - \Delta y_p^2 - 2x_p\Delta y_p + \\ &\quad +\Delta y_c) + x_c(-\Delta x_c - \Delta y_p) \end{aligned} \tag{23}$$

The calculation of α_M according to the definition in (15) is quite difficult. However, a bound of this function can be easily obtained. It follows from (23) that:

$$\begin{aligned} \alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) &\leq -\|x\|^2 + \|x\|(|\Delta x_c| + |\Delta y_p|) + \\ &+\|x\|((|\Delta x_c| + |\Delta y_p| + |\Delta y_p^2| + 2\|x\|\ |\Delta y_p| + |\Delta y_c|) \end{aligned}$$

Then, it results from (15) and the inequality above that:

$$\begin{aligned} \alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) \leq \sup_{\|x\|\geq 1} [-\|x\|^2 + \\ +\|x\|(2|\Delta x_c| + 2(\|x\| + 1)|\Delta y_p| + |\Delta y_p^2| + |\Delta y_c|)] \end{aligned}$$

Since outside the region D_2 the condition $\dot{V}(x) < -1$ is satisfied, the convergence speed s_1 (seventh step) can be chosen to be bounded between 0 and 1. Suppose that the choice is $s_1 = 0.5$. Then, the quantization must be chosen in order to satisfy $\alpha_M < -0.5$, condition that is verified using quantization intervals $\Delta q = \varepsilon = 0.07$ for all the variables.

If the restriction about the convergence speed is not taken into account and the goal is just assuring stability, that quantization interval of $\Delta q = 0.07$ is sufficiently small to guarantee convergence to the region given by $\|x\| < 0.4127$.

The simulation was done for an initial condition $x_p = 10$ and the results are shown in Figs. 5 to 7. The number of conversions performed by the A/D converter were 178 for 40 seconds of simulation time. The minimun time between two successive conversions was 5.6 milliseconds (at the begining of the simulation) while the maximum was greater than 2 seconds.

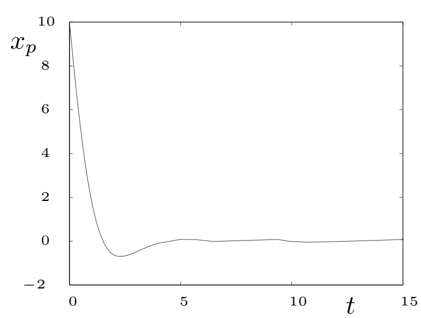

Figure 5: Plot of x_p for the plant with QSC

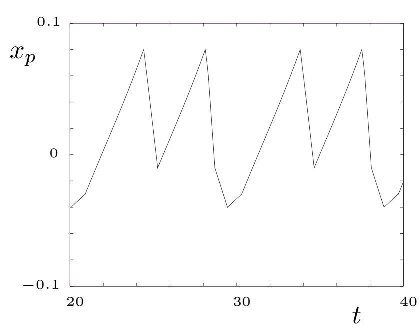

Figure 6: Final oscillations in x_p

The trajectory of Fig.5 seems to be quite similar to the one obtained with the continuous controller except for the final oscillations in Fig. 6. However, in Fig.7 the difference with the CCS behaviour is more evident.

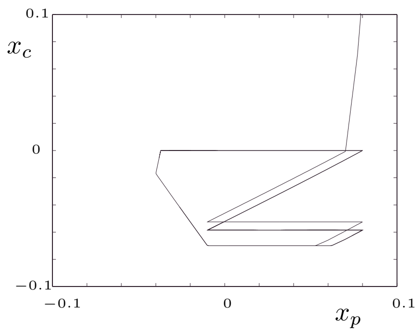

Figure 7: Final oscillations in the phase portrait

There, the phenomena of ultimately boundedness due to quantization and trajectory crossing due to hysteresis can be observed.

V. FURTHER ADVANTAGES OF QSC

While traditional discrete time controllers perform calculations at regular intervals, QSC controllers only do it when a variable becomes greater (or less) than some threshold. For instance, in the example of Section IV, when the trajectory arrives near the origin the controller performs about one calculation per second. Any discrete time controller using that sampling rate (and even a rate 10 times faster) will diverge for the same initial condition of $x_p = 10$ due to problems of *finite escape time*[6].

The following example illustrates better the reduction of the computational costs in the QSC scheme.

Consider the first order system

$$\dot{x} = \mathrm{sgn}(x) + u \tag{24}$$

Suppose that the controller can measure the variable x, but the A/D converter only produces even numbers (-2, 0, 2, 4, ...) giving the nearest to its analog input. When x is in the shaded region of Fig. 8 (between -1 and 1) the control system sees the value 0.

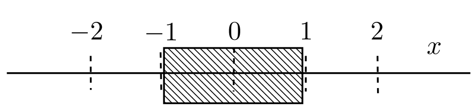

Figure 8: *Invisible* zone due to the quantization

Consider that the goal is keeping the value of x between -2 and 2. The time to go from 1 to 2 (or from -1 to -2) with $u = 0$ is $t_1 = 1$. This implies that it is impossible to find a discrete time controller achieving the proposed goal with a sampling period greater than t_1. The reason of this is that the controller cannot distinguish if the value of x is positive or negative when it is in the shaded region and then, the sign of u could be the same as the sign of x and the trajectory will abandon the desired region before the time t_1.

However, using QSC, the time between samplings can be done arbitrarily big. For instance, consider the following static control law

$$y_c = -u_c(1 + a) \tag{25}$$

where a is a positive constant. When x goes away of the shaded region the controller immediately detects the change and it inverts the sign of the derivative. The new speed on x is a. Then, x enters again the shaded region and the time to reach the origin is $1/a$. After that, x goes to the other bound of the shaded region, but with a new speed of $2+a$. When x leaves the shaded region again the controller inverts the sign of the derivative and we obtain a cyclic behaviour where x oscillates between 1 and -1. The time between successive A/D conversions is

$$t_2 = \frac{1}{a} + \frac{1}{2 + a} \tag{26}$$

It is clear that taking small values for the parameter a, t_2 can be done arbitrarily big. Then, with this implementation the number of calculations in the controller and the size of the oscillation can be considerably reduced.

Unfortunately, the QSC implementation is not exact. In fact, there are delays related to the presence of converters and the digital processor. In this last example, these delays must be smaller than the minimum sampling period $t_1 = 1$ in order to obtain the proposed goal of keeping x between 2 and -2.

One could think that if it is possible to implement a QSC achieving such minimum delay, then it would be possible to implement a discrete time controller using that sampling period. However, the delay in the QSC system is only the time required to detect the error and to perform the D/A conversion because the calculations can be done before the error is detected. This is possible because the QSC controller knows that its next input value can only adopt two different values[7].

A classic discrete time controller during each sampling period must perform the A/D conversion, calculate the next state and output of the controller and then perform the D/A conversion. The time required to finish all these tasks is much greater than the delay in the QSC scheme.

In (Kofman, 2001), further remarks about practical implementation of the converters and the delays in the QSC scheme can be found.

[6] The phrase "finite escape time" is used to describe the phenomenon that a trajectory escapes to infinity at a finite time (Khalil, 1996)

[7] In the example, if the last detected value of x was 0, then 2 and -2 are the only two values that the new input can adopt. Thus, the controller has two possible output values that could be calculated before the detection of the new input value. For a dynamic controller similar ideas can be applied.

VI. CONCLUSIONS

QSC constitutes a new class of digital controllers that can be designed taking into account the effects of D/A and A/D converters obtaining regional convergence of the trajectories to small regions around the equilibrium points and reducing the computational costs of the implementation.

A/D conversions are performed in a way that allows the controller having as much information as possible but without taking useless data. Similarly, D/A conversions are performed only when the corresponding variables change. In applications in which the optimization of conversions could be important (for instance in distributed control systems in which each conversion implies some transmission over a channel) the use of QSC might be highly convenient.

Although the principles of QSC implementation seems to be quite difficult (at least for people who is not familiar with the DEVS formalism) its programming is rather simple. Another advantage is the fact that for QSC design, classic design techniques for continuous controllers can be applied (for instance, in the example of Section IV, the original continuous controller was designed via *exact linearization* (Khalil, 1996)). Then, according to the design algorithm, an standard Lyapunov analysis to determine the regions of attraction and the ultimately bounds is performed. This kind of analysis is very common in nonlinear control systems design.

Future research in the line opened by this paper is pointed to the particularization of the methodology of QSC implementation for linear systems and for some classic control methods. Dealing with particular cases we expect to obtain closed formulas to calculate the quantization intervals. It is also important to obtain results related to closed loop performance measures and it would be interesting arriving to less conservative design conditions.

Some work must be also done in order to optimize the methods for real time simulation of DEVS taking into account the properties of QSC, since the minimization of the delays in the implementation is extremely important. The remarks of Section V about the fact that the controllers have some information about the next input event and the possibilities of precalculating the next output consitute a starting point in this direction.

Finally we should mention that the QSS represent just a small class of the systems that can be represented within the DEVS framework. In fact, any discrete time system (as standard discrete time controllers) can be also represented by a DEVS model. The possibility of finding new classes of DEVS models having nice properties for the control of continuous system seems to be more than promising.

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