Robust Stability Test of Polytopic Family of Polynomials: The Dixon's Resultant Method

Z.H. Wang†, H.Y. Hu† and T. Küpper‡

†*Institute of Vibration Engineering Research, Nanjing University of Aeronautics and Astronautics, 210016 Nanjing, P.R.China*
‡*Institute of Mathematics, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany*

Abstract— **This paper deals with the D-stability test of a polytope of polynomials when the boundary ∂D of a given simple connected domain D in the complex plan is described by a polynomial equation, a problem that covers two special but important cases: Hurwitz stability and Schur stability of a polytope of polynomials. Based on the "Edge Theorem" and the method of Dixon's resultant elimination, a new test approach is presented. By using the presented method, the stability test can be carried out by computing Dixon's resultants and solving linear matrix equations. Two examples are given to demonstrate the approach.**

Keywords— **polytope, polynomials, robust stability, Dixon's resultant.**

I. INTRODUCTION

The robust stability of dynamic systems has drawn great attention over the past decades since various uncertainties and errors always exist in the system modeling and parameter estimation. As is well-known, the stability analysis can be carried out by studying the root locations of the characteristic polynomials. Two important results for robust stability are due to Kharitonov (1979) who established a theory for the stability of interval families of polynomials, and to Bartlett *et al.* (1988) who developed the "Edge Theorem" for polytopic family of polynomials. Afterwards, different approaches, mainly based on the "Edge Theorem" or the "Zero-Exclusion Principle", for checking the robust stability of a given polytope of polynomials were developed. A comprehensive description of robust stability analysis under parametric uncertainty was given in (Bhattacharyya *et al.*, 1995).

The aim of this study is to present a new approach to test the D-stability of a given polytopic family Ω when the boundary ∂D of D is described by a polynomial equation. Here, a family Ω of polynomials is called D-stable if every polynomial in Ω is D-stable, namely, all the zeros of each member of Ω stay in D. The classical Hurwitz stability and Schur stability fall into this category. Result that is most closely related to this paper is (Zeheb, 1989). Zeheb's method is simple, but it cannot be applied to our case in general. Based on the "Edge Theorem", the idea of the Dixon's resultant for multivariate polynomials as in (Yang *et al.*, 1996) is applied in this paper to carry out the D-stability test. With this method, we need only to compute several Dixon's resultants and to solve some linear matrix equations. So the testing procedure is very simple.

II. PROBLEM FORMULATION

The polynomials under study are in the form

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \quad (1)$$

where the real coefficients $a_i,\ i = 0, 1, \cdots, n$, depend *linearly* on some parameters that vary in given intervals respectively. Then, the family of polynomials is a polytope, generated by a finite number of polynomials $p_1(\lambda),\ p_2(\lambda),\ \cdots,\ p_r(\lambda)$, as following

$$\Omega = \text{conv}\{p_1(\lambda),\ p_2(\lambda),\ \cdots,\ p_r(\lambda)\} \quad (2)$$

where $p_i(\lambda), (i = 1, 2, \cdots, r)$ are in the form of Eq.(1) and are called vertex polynomials. Let D be a simple connected domain in the complex plane with the boundary ∂D. The "Edge Theorem" states that the family Ω of polynomials is D-stable if and only if all the "exposed" one-dimensional edge polynomials are D-stable. Here, each edge polynomial generated by two vertex polynomials $p_i(\lambda)$ and $p_j(\lambda)$ is a sub-family $p_{ij}(\lambda, \mu)$ with a parameter μ

$$p_{ij}(\lambda,\ \mu) = (1-\mu)p_i(\lambda) + \mu p_j(\lambda), \quad \mu \in [0, 1] \quad (3)$$

In this paper, we assume that ∂D is described by a polynomial equation $b(x, y) = 0$. This covers two special but important cases. For the Schur stability of discrete-time systems, D is the open unit disk in the complex plane and $b(x, y) = x^2 + y^2 - 1$. For the Hurwitz stability of continuous-time systems, D is the open left half complex plane and $b(x, y) = x$. Since $p_{ij}(\lambda, \mu)$ is analytic with respect to λ and μ, a root $\lambda = \lambda(\mu)$ of $p_{ij}(\lambda, \mu)$ is continuous and cannot suddenly appear or disappear, or change its multiplicity at a finite point in the complex plane. With a variation of μ, thus, the sum of multiplicity of all roots of $p_{ij}(\lambda, \mu) = 0$ in D^c, the complement of set D, can change only if a root appears on or crosses the boundary ∂D.

Let $\lambda = x+\mathrm{i}y$, we see that the edge polynomial is marginal stable if and only if (x, y, μ) is a real common root of three polynomials $\mathrm{Re}[p_{ij}(x+\mathrm{i}y,\mu)]$, $\mathrm{Im}[p_{ij}(x+\mathrm{i}y,\mu)]$, and $b(x,y)$. Thus, $p_{ij}(x+\mathrm{i}y,\mu) \neq 0$, subject to $\mu \in [0,1]$ and $b(x,y) = 0$, holds true if and only if $\mathrm{Re}[p_{ij}(x+\mathrm{i}y,\mu)] = 0$, $\mathrm{Im}[p_{ij}(x+\mathrm{i}y,\mu)] = 0$ and $b(x,y) = 0$ have no real common roots for all $\mu \in [0,1]$. This leads to the following

Theorem 1 *The polytopic family Ω of polynomials is D-stable if and only if (a): at least one of vertex polynomials has all roots in D; (b): for each "exposed" edge polynomial, $Re[p_{ij}(x+iy,\mu)] = 0$, $Im[p_{ij}(x+iy,\mu)] = 0$ and $b(x,y) = 0$ have no real common real roots (x,y) for all $\mu \in [0,1]$, where $p_{ij}(\lambda,\mu)$ is defined in Eq.(3).*

We are interested in determining all the *possible* values of $\mu \in [0,1]$ for which the three polynomials $\mathrm{Re}[p_{ij}(x+\mathrm{i}y,\mu)]$, $\mathrm{Im}[p_{ij}(x+\mathrm{i}y,\mu)]$ and $b(x,y)$ have real common roots, since the corresponding polynomials consist of the test set of polynomials. These values of μ can be determined by using the theories of polynomials such as Sylvester resultant (Yang *et al.*, 1996), the method of Gröbner basis (Adams and Loustaunau, 1994). In this paper, we use the idea of the Dixon's resultant to complete the D-stability analysis. The D-stability can be carried out by simply determining whether the linear matrix equations for the edge polynomials have real solutions subject to some very simple conditions.

III. STABILITY TEST

For each "exposed" edge, we denote by $f_1(x,y)$, $f_2(x,y)$ the real and imaginary parts of $p_{ij}(x+\mathrm{i}y,\mu)$ respectively, and $f_3(x,y) = b(x,y)$, the boundary polynomial. The peculiarity of our method is to consider these three polynomials in three unknowns x, y, and μ as to three polynomials in two unknowns x and y, with μ as a parameter. Namely, we study the following set of polynomial equations

$$PS: f_1(x,y) = 0, \quad f_2(x,y) = 0, \quad f_3(x,y) = 0 \quad (4)$$

The main procedures used here are as follows. First, a set of polynomial equations is constructed from PS and the new polynomial equations are considered as to linear equations with respect to the different powers of x and y. Then, solving PS is studied on the basis of theory of linear algebraic equations.

A. The Method of Dixon's Resultant

Let us first consider two n-th order real polynomials $f(x)$ and $g(x)$, the expression

$$\delta(x,\alpha) = \frac{f(x)g(\alpha) - f(\alpha)g(x)}{x-\alpha} \quad (5)$$

can be easily verified to be a polynomial with respect to x, α. We assume that different terms have different powers with respect to α, as well as with respect to x. For any common root x_0 of $f(x)$ and $g(x)$, we have $\delta(x_0,\alpha) = 0$ for all α. Thus the coefficients of δ with respect to the powers $\alpha^n, \alpha^{n-1}, \cdots, \alpha, \alpha^0(=1)$ must vanish. In terms of the powers $x^n, x^{n-1}, \cdots, x, x^0(=1)$, the vanishing coefficients can be expressed as to a linear matrix equation. The coefficient matrix of the linear matrix equation is called Bezout matrix, and its determinant is called the Bezout resultant, which is the same as the Sylvester resultant but the Bezout matrix has lower oder than the corresponding Sylvester matrix. It is well-known that $f(x)$ and $g(x)$ has a common nonconstant factor if (and only if) the Bezout resultant is zero (Yang *et al.*, 1996). If $\deg(f) = n > m = \deg(g)$, then the above process can be performed by writing $g(x) = 0\cdot x^n + 0\cdot x^{n-1} + \cdots + 0\cdot x^{m+1} + g(x)$.

As for PS, let α, β be two new variables, and define a 3rd order determinant for PS.

$$\Delta(x,y;\alpha,\beta) = \begin{bmatrix} f_1(x,y) & f_2(x,y) & f_3(x,y) \\ f_1(\alpha,y) & f_2(\alpha,y) & f_3(\alpha,y) \\ f_1(\alpha,\beta) & f_2(\alpha,\beta) & f_3(\alpha,\beta) \end{bmatrix} \quad (6)$$

Since $\Delta(\alpha,y;\alpha,\beta) = 0$ and $\Delta(x,\beta;\alpha,\beta) = 0$, there is a factor $(x-\alpha)(y-\beta)$ in $\Delta(x,y;\alpha,\beta)$, thus

$$\delta(x,y;\alpha,\beta) = \frac{\Delta(x,y;\alpha,\beta)}{(x-\alpha)(y-\beta)} \quad (7)$$

is a polynomial with respect to x, y, α and β, and is called the Dixon's reduced polynomial of PS.

If we expand the three polynomials in PS to be

$$f_k(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}^{(k)} x^i y^j, \quad (k = 1,2,3) \quad (8)$$

(here, we remember that μ is considered to be a parameter), then we can rewrite δ as to

$$\delta(x,y;\alpha,\beta) = \sum_{k=0}^{2m-1}\sum_{l=0}^{n-1}\sum_{i=0}^{m-1}\sum_{j=0}^{2n-1} d_{ijkl} x^i y^j \alpha^k \beta^l \quad (9)$$

As in the above simple case, each real common root (x_0, y_0) of PS satisfies also the following equations:

$$\sum_{i=0}^{m-1}\sum_{j=0}^{2n-1} d_{ijkl} x_0^i y_0^j = 0 \quad (10)$$

for all $k = 0,1,2,\cdots,2m-1$ and $l = 0,1,2,\cdots,n-1$. These equations are called the Dixon's reduced polynomial equations, and can be rewritten in the form of linear matrix equation

$$\mathbf{Mz} = \mathbf{0}, \ \mathbf{z} = \begin{bmatrix} x^{2m-1}y^{n-1} & \cdots & y & x & 1 \end{bmatrix}^T \quad (11)$$

with $\mathbf{M} \in R^{2mn\times 2mn}$. The determinant $J(\mu) = |\mathbf{M}|$ is called the Dixon's resultant. If $f_1(x,y)$, $f_2(x,y)$ and $f_3(x,y)$ have a real common root, the above linear matrix equation must have a non-zero solution since $\mathbf{z} \neq 0$, hence $J(\mu) = 0$.

Generally, the following two cases are possible:

- The order of $\mathbf{M}$ may be smaller than $2mn$, since it is not necessary that all the powers $x^\alpha y^\beta$ up to $x^m y^n$ appear in (8).
- $\mathbf{M}$ may not be square. If we add some formal "zero" coefficients to the equations, the determinant $|\mathbf{M}|$ vanish, then the resultant does not give any information for determining the possible values μ.

In (Kapur *et al.*, 1994), it has been shown that there exists a nonzero necessary condition $\tilde{J}(\mu) = 0$ for the existence of a common root of a generally given set of polynomial equations, and an algorithm is provided for computing such a nonzero necessary condition, which has successfully been implemented in *MAPLE*, a well-known computer algebra.

For our purpose, however, the linear matrix equation (11) for PS is usually enough. What is important is that one can transform the coefficient matrix $\mathbf{M}$ into another matrix of triangular form, then the linear matrix equation can be solved recursively. Usually, one can solve two of the simplified linear equations for (x, y) and then substitute it into a third equation, or the boundary condition, to obtain a necessary condition $\tilde{J}(\mu) = 0$, where $\tilde{J}(\mu)$ is a polynomial. For simplicity, $\tilde{J}(\mu)$ is also called Dixon's resultant. At each root $\mu \in [0, 1]$ of $J_{ij}(\mu)$ or $\tilde{J}_{ij}(\mu)$, one can solve easily the corresponding linear equation (11). We call the above procedure to be the Dixon's resultant method.

B. D-stability Test

From the above analysis, we see that the D-stability of $p_{ij}(\lambda, \mu)$ can be changed only if $\mu \in [0, 1]$ reaches the zeros of $J_{ij}(\mu)$ or $\tilde{J}_{ij}(\mu)$ with a variation of μ from 0 to 1. Once a non-zero Dixon's resultant $J_{ij}(\mu)$ or $\tilde{J}_{ij}(\mu)$ is obtained, the test set of polynomials that govern the D-stability of the whole family can be achieved. In fact, let $\Lambda_{ij} = \{\mu \in [0, 1] : \ J_{ij}(\mu) = 0, \text{ or } \tilde{J}_{ij}(\mu) = 0\}$, $T_{ij} = \{p_{ij}(\lambda, \mu) : \ \mu \in \Lambda_{ij}, \ \mu = 0, 1\}$, and $T = \bigcup_{i,j} T_{ij}$, then, it is obvious that the polytopic family is D-stable if and only if T is D-stable. Since the number of exposed edges and all the numbers of zeros of $J_{ij}(\mu)$ or $\tilde{J}_{ij}(\mu)$ are finite, the number of testing polynomials is finite.

In practice, we need not to test the D-stability of each member of T, but to solve some linear matrix equations only. To see this, we note that PS has no real common solution if and only if for each edge polynomial, one of the following three cases occurs:

(C1) $J_{ij}(\mu)$ or $\tilde{J}_{ij}(\mu)$ has no real zeros in $\mu \in [0, 1]$;

or at each $\mu \in \Lambda_{ij}$,

(C2) Eq. (11) has no real solution $\mathbf{z}$; or

(C3) Eq. (11) has a solution $\mathbf{z_0}$ which gives a pair values of $x = x_0$ and $y = y_0$ but it is not compatible to the powers in $\mathbf{z} = \mathbf{z_0}$.

It concludes to the following theorem.

Theorem 2 *The polytopic family Ω, defined in (2), of polynomials is D-stable if and only if (a): at least one of the vertex polynomials is D-stable; (b): for each "exposed" edge, one of the conditions, (C1), (C2) or (C3), holds true.*

This theorem yields a simple algorithm as below to test the D-stability of Ω when ∂D can be described by $b(x, y) = 0$.

Algorithm 1 *D-Stability Test of Ω*

- **Step 1:** *Select one of the vertex polynomials, say, $p_1(\lambda)$, and test whether all of its zeros lie in D. If it has a root in D^c, then Ω is not D-stable. Otherwise, go to the next step.*
- **Step 2:** *For an exposed edge polynomial generated by $p_1(\lambda)$ and another vertex polynomial, say $p_2(\lambda)$, compute the corresponding Dixon's resultant $J_{12}(\mu)$ or $\tilde{J}_{12}(\mu)$. If there exists a real root $\mu \in [0, 1]$ of $J_{12}(\mu)$ or $\tilde{J}_{12}(\mu)$ such that the corresponding equation (11) has a real solution for which the PS has a real common root, then Ω is not D-stable. Otherwise, the edge polynomial $p_{12}(\lambda)$ is D-stable, and go to the next step.*
- **Step 3:** *Repeat step 2 for each exposed edge polynomial. If there is a edge polynomial that is not D-stable, then Ω is not D-stable, otherwise Ω is D-stable.*

C. Illustrative Examples

Now, two simple examples are given to demonstrate the new approach. Since the testing set for the robust Hurwitz stability can be obtained easily by using the Sylvester resultant, the following two examples are all about robust Schur stability. The boundary polynomial is now $b(x, y) = x^2 + y^2 - 1$. For other discussion on Schur stability, it is referred to see (Ackermann and Barmish, 1988) and (Kraus *et al.*, 1992).

Example 1 Consider first a simple polynomial $p(\lambda) = \lambda^2 + a_1\lambda + a_2$. It is Schur stable when $a_1 = -21/20$, $a_2 = 27/100$. If a_1 is assumed to have $\pm 20\%$ variation, we have two Schur stable polynomials $p_1(\lambda) = \lambda^2 - \frac{63}{50}\lambda + \frac{27}{100}$ and $p_2(\lambda) = \lambda^2 - \frac{21}{25}\lambda + \frac{27}{100}$: the two real zeros of $p_1(\lambda)$ are 0.9862 and 0.2738, and the absolute of the two conjugate zeros of $p_2(\lambda)$ is 0.5196. We study the robust Schur stability of

$$\Omega = \{p_{12}(\lambda, \mu) = (1 - \mu)\, p_1 + \mu\, p_2 : \ \mu \in [0, 1]\} \quad (12)$$

and need only to check whether the family $p_{12}(\lambda, \mu)$ has no roots on ∂D for all $\mu \in [0, 1]$. Straightforward computation gives the Dixon's resultant of the real and imaginary parts $R(x, y, \mu)$, $S(x, y, \mu)$ of $p_{12}(x+\mathrm{i}y, \mu)$ and $b(x, y)$, with respect to x, y, as follows

$$J(\mu) = (42\mu + 1)(42\mu - 253) \quad (13)$$

which is determined except for a non-zero factor. Since $J(\mu)$ has no real roots in $\mu \in [0,\ 1]$, all the roots of $p_{12}(\lambda,\mu)$ stay in the open unit disk D. Thus, the polytope Ω is robust Schur stable.

Example 2 We consider the robust Schur stability of Ω in (12) with $p_1(\lambda) = \lambda^3 - \frac{47}{60}(1+r)\lambda^2 + \frac{1}{5}\lambda - \frac{1}{60}$ and $p_2(\lambda) = \lambda^3 - \frac{47}{60}(1-r)\lambda^2 + \frac{1}{5}\lambda - \frac{1}{60}$, where $r = 20\%$. It is also easy to know that the two vertex polynomials are Schur stable. What follows is to check whether all the zeros of $p_{12}(\lambda,\mu)$ fall into D for all $\mu \in [0,1]$.

A direct computation shows that $\delta(x,y;\alpha,\beta)$ contains 11 terms with respect to the powers of α and β, with two terms proportional to $b(x,y)$. In addition, a common factor y exists in some Dixon's reduced polynomials. So, the case $y=0$, $x=\pm 1$ must be considered. If there exists some $\mu \in [0,1]$ such that $R(1,0,\mu)=0$, $S(1,0,\mu)=0$; or $R(-1,0,\mu)=0$, $S(-1,0,\mu)=0$, then the polytope is not robust Schur stable. It is easy to know that this is not the case. We eliminate the apparent redundant equation and the common factor y, and rewrite the Dixon's reduced polynomial equations as follows

$$\mathbf{Mz} = \mathbf{0},\ \mathbf{z} = [\ y^4 \quad x^2y^2 \quad xy^2 \quad y^2 \quad x^2 \quad x \quad y \quad 1\ ]^T \tag{14}$$

By using the method of Gauss elimination, $\mathbf{M}$ is changed into the following form

$$\mathbf{U} = \begin{bmatrix} * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14677+94\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

then, Eq.(15) and Eq.(14) give two equations

$$-14677 + 94\mu = 0, \quad -x + 24 = 0 \tag{16}$$

which has no solutions $\mu \in [0,1]$ and $x \in [-1,1]$. So $p_{12}(x+\mathrm{i}y,\mu) \neq 0$ holds true for all $\mu \in [0,1]$. As a result, Ω is robust Schur stable.

IV. DISCUSSIONS

The Dixon's resultant method is used in this paper not only for determining the possible values of $\mu \in [0,1]$ that render the edge polynomials marginal stable, but also in checking the D-stability by simply solving some linear matrix equations. This is the main advantage of the present approach.

There are some other choices for determining the possible values of $\mu \in [0,1]$ that render the edge polynomials marginal stable, if we are interested in only the test set of polynomials. For example, let $\mathrm{Res}(f,g,x)$ denote the resultant of f and g with respect to x, we define $u_1(y,\mu) = \mathrm{Res}(f_1,f_2,x)$, $u_2(y,\mu) = \mathrm{Res}(f_1,b,x)$ and $J(\mu) = \mathrm{Res}(u_1,u_2,y)$. Every $\mu \in [0,1]$ for which PS has real common roots must be a root of $J(\mu)$. Hence, the possible values of $\mu \in [0,1]$ that may destroy the stability of the edge families can only be the roots of $J_i(\mu)$. Using of Sylvester resultant, however, does not lead to an effective testing procedure as done above by using the Dixon's resultant method.

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